3264 & All That
Intersection Theory in Algebraic Geometry

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Chapter 1

Solutions to Selected Exercises

1.1 Chapter 1

Exercise 1.1. **??** Let $Y \subset \mathbb{A}^n$ be a subvariety not containing the origin, and let $W \subset \mathbb{P}^1 \times \mathbb{A}^n$ be the closure of the locus

$$W^\circ = \{(t, z) \mid z \in t \cdot Y\}$$

as in the proof of Prosposition **??**. Show that the fiber of W over $t = 0$ is the cone with vertex the origin $0 \in \mathbb{A}^n$ over the intersection $\overline{Y} \cap H_\infty$, where $\overline{Y} \subset \mathbb{P}^n$ is the closure of Y in $\mathbb{P}^n$ and $H_\infty = \mathbb{P}^n \setminus \mathbb{A}^n$ is the hyperplane at infinity.

Solution to Exercise ??: Let us denote by W_0 the fiber of W over $t = 0$. Let $Y = V(f_1, \ldots, f_m)$ where

$$f_i = f_{i,0} + f_{i,1} + \ldots + f_{i,d_i-1} + f_{i,d_i}$$

and $f_{i,j}$ is a homogeneous degree j polyonomials in n variables $z_1, \ldots, z_n$. To obtain $\overline{Y} \subset \mathbb{P}^n$, we need to homogenize all polynomials f_i using a further variable z_0, getting polynomials

$$\overline{f}_i = z_0^{d_i} f_{i,0} + z_0^{d_i-1} f_{i,1} + \ldots + z_0 f_{i,d_i-1} + f_{i,d_i},$$

so that $\overline{Y} = V(\overline{f}_1, \ldots, \overline{f}_m)$. Intersecting with H_∞ means just to set $z_0 = 0$, so that we get $\overline{Y} \cap H_\infty = V(z_0, \overline{f}_1, \ldots, \overline{f}_m) = V(z_0, f_{1,d_1}, \ldots, f_{m,d_m})$; in $\mathbb{A}^n$, the cone over it is just $V(f_{1,d_1}, \ldots, f_{m,d_m})$; let us now show that this is indeed W_0. From the definition of W°, we have that on the fiber over $t \neq 0$ it vanishes every polynomial of the kind

$f_i(\frac{z_1}{t}, \ldots, \frac{z_n}{t})$, that means, (multiplying by the right power of t), on the fiber over t it vanishes the polynomial

$$t^{d_i} f_{i,0} + t^{d_i - 1} f_{i,1} + \ldots + t f_{i,d_i - 1} + f_{i,d_i}.$$

Considering these as polynomials on the whole $\mathbb{P}^1 \times \mathbb{A}^n$, they have to vanish on W (because they vanish on a dense subset), so setting $t = 0$ we have that W_0 is contained in $V(f_{1,d_1}, \ldots, f_{m,d_m})$.

To prove that W_0 contains the cone $V(f_{1,d_1}, \ldots, f_{m,d_m})$ (and so that they are indeed equal). First, let us show that W_0 is indeed a cone; in fact, this follows from the fact that given any nonzero scalar λ the linear transformation

$$(t, z) \mapsto (\lambda t, \lambda \cdot z)$$

does indeed keep fixed W° (and hence W and W_0), because W° is just the set of all (t, z) such that $t^{-1} z \in Y$; but this transformation on W_0 is just the rescaling by λ, so W_0 is invariant under such rescalings. Then, let us consider the closure $\overline{W}$ of W° in $\mathbb{P}^1 \times \mathbb{P}^n$; this does contain the (not closed) subvariety $(\mathbb{A}^1 \setminus 0) \times (\overline{Y} \cap H_\infty)$ (because all fibers of W° have the same "limit at infinity"), so that also the fiber $\overline{W}_0$ over $t = 0$ of $\overline{W}$ contains $\overline{Y} \cap H_\infty$.

All of this proves that the limit at infinity of W_0 is *contained* in $\overline{Y} \cap H_\infty$, that means, $\overline{W}_0$ is the cone over a subvariety $Z \subset Y \cap H_\infty$. Now, $\overline{W}$ is irreducible of dimension $dim(Y) + 1$, and $\overline{W}_0$ is defined by the equation $t = 0$, so by Theorem **??** $\overline{W}_0$ is purely dimensional of dimension $dim(Y)$. Now, if Z is not the whole $Y \cap H_\infty$, then $Y \cap H_\infty$ is an irreducible component of $\overline{W}_0$, and it has dimension $dim(Y) - 1$ and this is an absurd. □

Exercise 1.2. **??** Show that if X is an irreducible plane cubic with a node, then $c_1 : Pic(X) \to A_1(X)$ is not a monomorphism as follows. Show that there is no birational map from X to $\mathbb{P}^1$. Use this to show that if $p \neq q \in X$ are smooth points then the line bundles $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ are non-isomorphic. Show, however, that the zero loci of their unique sections, the points p and q, are rationally equivalent.

Solution to Exercise ??: Let $\mathbb{P}^1 = X^\nu \xrightarrow{\nu} X$ be the normalization of X, a smooth rational curve, and let n_1 and n_2 be the points that maps down to the node n in X. In $\mathbb{P}^1$ the preimages of points p and q (that we will still call p and q) are rationally equivalent; in fact, the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ realizes the rational equivalence (picking $t_0 = p$ and $t_1 = q$). Applying ν to the second factor, we get $(Id_{\mathbb{P}^1} \times \nu)(\Delta) \subset \mathbb{P}^1 \times X$, that realizes the equivalence between p and q in X. To prove that $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ are non-isomorphic, we need to prove that $\mathcal{O}_X(p) \otimes \mathcal{O}_X(q)^* \cong \mathcal{O}_X(p - q)$ has no nonvanishing sections (actually, we will see that it has no *nonzero* section). A section of $\mathcal{O}_X(p - q)$ would be a meromorphic function having a simple pole in p and a simple zero in q, that is, a morphism $f : X \to \mathbb{P}^1$ such that $f^{-1}(0) = q$ and $f^{-1}(\infty) = p$.

This would pull back by the normalization map to a morphism $f^\nu : \mathbb{P}^1 \to \mathbb{P}^1$ such that $(f^\nu)^{-1}(0) = q$ and $(f^\nu)^{-1}(\infty) = p$ (this holds scheme-theoretically because zero and pole are simple), so that f^ν should be of degree 1, that is, an isomorphism; this is an absurd though, because f^ν would map the two distinct points n_1 and n_2 to the same point. Bonus question: are $\mathcal{O}_X(2p)$ and $\mathcal{O}_X(2q)$ are isomorphic? □

Exercise 1.3. **??** Show that if $X \subset \mathbb{P}^3$ is a quadric cone with vertex p then $A_1(X) = \mathbb{Z}$, generated by the class $[L]$ of a line $L \subset X$, and show that the image of $c_1 : Pic(X) \to A_1(X)$ is $2\mathbb{Z}$. Hint: show that there is no line bundle on X with first Chern class $[L]$ by considering the degree of its restriction to L (see Example **??**).

Solution to Exercise ??: The degree in the ambient space $\mathbb{P}^3$ gives a surjective homomorphism $A_1(X) \to \mathbb{Z}$ (this can also be seen as the pushforward $A_1(X) \to A_1(\mathbb{P}^3) \cong \mathbb{Z}$); let's prove now the injectivity too, in particular, proving that every degree d reduced curve C in X is rationally equivalent to the union of d distinct lines through the vertex; in this way, it is easy to prove that all degree d curves in X are rationally equivalent. We will use exercise **??** to explicitly describe the equivalence; let H be an hyperplane in $\mathbb{P}^3$ not containing the vertex p, and having d distinct points of intersection with C; let's consider now the affine space $\mathbb{P}^3 \setminus H$, with coordinates such that p is in the origin. Now, by exercise **??**, in this coordinates the family tC for $t \to 0$ deforms C into a cone over $C \cap H$, that is, d lines through the vertex (note that the whole transition happens inside X).

To prove that $[L]$ is not in the image of $Pic(X)$, let's suppose there exists a line bundle $\mathcal{L}$ with a section having a simple zero along a line L of the ruling; this bundle satisfies $\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X(H)$ where $\mathcal{O}_X(H)$ is the line bundle of the hyperplane section in $\mathbb{P}^3$, because any union of two lines in the ruling is an hyperplane section. Let's consider now the restriction (pullback) $\mathcal{L}|_L$; from the relation above, in $A_0(L) = \mathbb{Z}$ we would have

$$2deg(c_1(\mathcal{L}|_L)) = deg(c_1(\mathcal{O}_L(H))) = 1$$

that leads to an absurd because the Chern class belongs to $A_0(L) = \mathbb{Z}$. □

1.2 Chapter 2

Exercise 1.4. **??** Let $\nu = \nu_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ be the quadratic Veronese map. If $C \subset \mathbb{P}^2$ is a plane curve of degree d, show that the image $\nu(C)$ has degree $2d$. (In particular, this means that the Veronese surface $S \subset \mathbb{P}^5$ contains only curves of even degree!) More generally, let $\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ be the degree d Veronese map. If $X \subset \mathbb{P}^n$ is a variety of dimension k and degree e, show that the image $\nu(X)$ has degree $d^k e$.

Solution to Exercise ??: Let's apply the push-pull formula **??** (**??**). Let $\zeta_N \in A^1(\mathbb{P}^N)$ be the hyperplane class in $\mathbb{P}^N$, and $\zeta_n \in A^1(\mathbb{P}^n)$ be the hyperplane class in $\mathbb{P}^n$; we know that $\nu^*\zeta_N = d\zeta_n$. Let $[X] \in A^{n-k}(\mathbb{P}^n)$ be the class of X: the degree of $\nu(X)$ is given by the class $\zeta_N^k \cdot \nu_*([X]) \in A^N(\mathbb{P}^N)$, and by the push pull formula we get

$$deg(\zeta_N^k \cdot \nu_*([X])) = deg(\nu_*((\nu^*\zeta_N)^k \cdot [X])) = d^k deg(\zeta_n^k \cdot [X]) = d^k e.$$

□

Exercise 1.5. **??** Let $\sigma = \sigma_{r,s} : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$ be the Segre map, and let $X \subset \mathbb{P}^r \times \mathbb{P}^s$ be a subvariety of codimension k. If the class $[X] \in A^k(\mathbb{P}^r \times \mathbb{P}^s)$ is given by

$$[X] = c_0\alpha^k + c_1\alpha^{k-1}\beta + \cdots + c_k\beta^k$$

(where α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ are the pullbacks of the hyperplane classes, and we take $c_i = 0$ if $i > s$ or $k - i > r$),

(a) Show that all $c_i \geq 0$.
(b) Calculate the degree of the image $\sigma(X) \subset \mathbb{P}^{(r+1)(s+1)-1}$; and, using this and the first part,
(c) Show that any linear space $\Lambda \subset \Sigma_{r,s} \subset \mathbb{P}^{(r+1)(s+1)-1}$ contained in the Segre variety lies in a fiber of a projection map.

Solution to Exercise ??: About part (a), this coefficients appear as intersection products with classes $\alpha^{r-i}\beta^{s-j}$ with $i + j = k$ corresponding to products of subspaces $\mathbb{P}^i \times \mathbb{P}^j$; in characteristic zero, we can apply Kleiman's theorem, and find that the coefficients arise as zero-dimensional transverse intersection, that is, a nonnegative number of points. In nonzero characteristic, with a further step one can prove that the intersection with the general k-dimensional product of subspaces is zero dimensional (otherwise, X would be higher dimensional); so, we can use the fact that a dimensionally transverse intersection always gives a nonnegative number. About part (b), we can use the push-pull formula; let $\zeta = \zeta_{rs+r+s}$ the hyperplane class in $\mathbb{P}^{rs+r+s}$: then we have

$$deg(\zeta^{r+s-k} \cdot \sigma_*[X]) = deg((\alpha + \beta)^{r+s-k} \cdot [X]) = \sum_{i=0}^{k} \binom{r+s-k}{s-i} c_i.$$

A linear space has degree 1; so, we need to find when this number achieves 1, and this (given part (a)) is when there is only one nonzero coefficient, and with the binomial coefficient being one: the only possibilities are then $\alpha^r\beta^{k-r}$ or $\alpha^{k-s}\beta^s$ (and they happen only if k is greater than r and s), that is, linear subspaces of fibers, that completes part (c). □

Exercise 1.6. **??** Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map given by

$$\varphi : (x_0, x_1, x_2) \mapsto \left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right),$$

or, equivalently,

$$\varphi : (x_0, x_1, x_2) \mapsto (x_1 x_2, x_0 x_2, x_0 x_1)$$

and let $\Gamma_\varphi \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of φ. Find the class

$$[\Gamma_\varphi] \in A^2(\mathbb{P}^2 \times \mathbb{P}^2).$$

Solution to Exercise ??: The class is $[\Gamma_\varphi] = \alpha^2 + 2\alpha\beta + \beta^2$. To prove it, let us use the indeterminate coefficients method; the class we are looking for is of the kind

$$c_0\alpha^2 + c_1\alpha\beta + c_2\beta^2.$$

Note that φ is not regular, so we can't use Proposition **??**; from the second expression (with double products) we can see that is defined only away from points with two zero coordinates. This morphism is birational though: from the first expression with reciprocals, it's easy to see that it is actually an involution. So, the general intersection with both fibers in $\mathbb{P}^2 \times \mathbb{P}^2$ is one reduced point, giving $c_0 = c_2 = 1$. To find c_1, we have to intersect with a general product $\mathbb{P}^1 \times \mathbb{P}^1$, that is, restricting φ to a general line in $\mathbb{P}^2$, and asking for the intersection of the image with a general line. Restricting φ to a general line we get a regular morphism of degree 2, so that the image is a smooth conic in $\mathbb{P}^2$, whose intersection with a general line is transverse and given by 2 points, so that $c_1 = 2$. □

Exercise 1.7. **??** Let $\sigma : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ be the Segre map. Find the class of the graph of σ in $A(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^8)$.

Solution to Exercise ??: Let α, β, ζ be the hyperplane classes in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^8$. The graph is 4 dimensional, so its class is going to be of the form

$$\sum_{\substack{0 \le j \le 2 \\ 0 \le i \le 2}} c_{ij}\alpha^i \beta^j \zeta^{8-i-j}.$$

To find the coefficients c_{ij}, we have to intersect with a general $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^{8-i-j}$; in particular, the degree of this intersection is going to be the degree of σ restricted to $\mathbb{P}^i \times \mathbb{P}^j$; these are just degrees of smaller Segre embeddings, that is, smaller binomial coefficients $\binom{i+j}{i}$. To see that the intersection is trasverse, first note that the intersection with any $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^8$ is transverse because this is a graph; then, the intersection with a general $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^{8-i-j}$ is transverse, because of Bertini's theorem in $\mathbb{P}^8$. The class is then

$$\sum_{\substack{0 \le j \le 2 \\ 0 \le i \le 2}} \binom{i+j}{i} \alpha^i \beta^j \zeta^{8-i-j}.$$

□

Exercise 1.8. **??** Let $s : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^{2*}$ be the rational map sending $(p, q) \in \mathbb{P}^2 \times \mathbb{P}^2$ to the line $\overline{p, q}$. Find the class of the graph of s in $A(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{2*})$.

Solution to Exercise ??: Note at first that the closure of the graph is exactly the locus

$$\Psi = \{(p, q, L) | p, q \in L\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{2*}$$

since this is closed and irreducible (looking at the projection onto $\mathbb{P}^{2*}$) and contains the graph as open subset. Let now α, β, γ be the hyperplane classes in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{2*}$. The class we are looking for is of the form

$$[\Psi] = c_{00}\gamma^2 + c_{10}\alpha\gamma + c_{01}\beta\gamma + c_{20}\alpha^2 + c_{11}\alpha\beta + c_{02}\beta^2.$$

Assuming transversality (by Kleiman's theorem, or a direct evaluation of tangent spaces) we get

$$c_{00} = \#(\Psi \cap \{p\} \times \{q\} \times \mathbb{P}^{2*}) = \#\{(p, q, \overline{pq})\} = 1$$

$$c_{10} = \#(\Psi \cap \mathbb{P}^1 \times \{q\} \times \mathbb{P}^{1*}) = 1$$

$$c_{01} = \#(\Psi \cap \{p\} \times \mathbb{P}^1 \times \mathbb{P}^{1*}) = 1$$

$$c_{20} = \#(\Psi \cap \mathbb{P}^2 \times \{q\} \times \{L\}) = 0$$

$$c_{11} = \#(\Psi \cap \mathbb{P}^1 \times \mathbb{P}^1 \times \{L\}) = 1$$

$$c_{02} = \#(\Psi \cap \{p\} \times \mathbb{P}^2 \times \{L\}) = 0$$

where every calculation comes from easy geometric observations. The class is then

$$[\Psi] = \gamma^2 + \alpha\gamma + \beta\gamma + \alpha\beta.$$

This could have been seen also from the fact that Ψ is the intersection of the loci $\{p \in L\}$ and $\{q \in L\}$ whose classes are $\gamma + \alpha$ and $\gamma + \beta$. □

Exercise 1.9. **??** Let $X_1, \ldots, X_n \subset \mathbb{P}^n$ be hypersurfaces of degrees $d_1, \ldots, d_n$. Let $p \in \mathbb{P}^n$ be a point, and suppose that the hypersurface X_i has multiplicity m_i at p; suppose moreover that the intersection of the projective tangent cones $\mathbb{P}TC_pX_i$ to X_i at p is empty. Use the description of the Chow ring of the blow-up of $\mathbb{P}^n$ at p to show that the number of points of intersection of the X_i away from p is

$$\#\left(\bigcap (X_i \setminus \{p\})\right) = \prod d_i - \prod m_i.$$

Solution to Exercise ??: Let B be the blow up of $\mathbb{P}^n$ at p. The proper transforms $X'_1, \ldots, X'_n$ of the hypersurfaces $X_1, \ldots, X_n$ in B intersect the exceptional divisor E in their tangent cones at p, that are disjoint; thus, the X'_i don't intersect inside E; their intersection in B correspond exactly to the intersection of the X_i in $\mathbb{P}^n$ away from p;

now, the class of X_i' is given by the relation $[X_i'] + m_i[E] = \pi^*[X_i]$, so that following the notation in **??** we get $[X_i'] = d_i\lambda - m_i e$, whose degree of intersection is $\prod d_i - \prod m_i$; to prove that they intersect in exactly this number of points, in charasteristic 0 by Kleiman's theorem in B this is immediately true. Otherwise, one way is to prove that among the n-tuples $(X_1, \ldots, X_n)$ of hypersurfaces with the right degrees and multiplicities at p, the n-tuples whose intersection away from p is not transverse are few (less dimensional than the space of all n-tuples) so that in the general case the intersection is indeed transverse, or to use a refinement of Bertini's theorem. □

Exercise 1.10. **??** Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Suppose that X has an ordinary double point (that is, a point $p \in X$ such that the projective tangent cone $\mathbb{P}TC_pX$ is a smooth quadric) and is otherwise smooth. What is the degree of the dual hypersurface $X^* \subset \mathbb{P}^{n*}$?

Solution to Exercise ??: Let's work in the blow up B of $\mathbb{P}^n$ at p, with X' the proper transform of X, whose class in $A^*(B)$ is $d\lambda - 2e$. The rational map $\mathbb{P}^n - \to \mathbb{P}^{n*}$ given by partial derivatives of a function f defining X extend to a regular morphism $B \to \mathbb{P}^{n*}$, and by this map the hyperplane section of $\mathbb{P}^{n*}$ pulls back to the class $(d-1)\lambda - e \in A^1(B)$, because the morphism is given by degree $d-1$ polynomials all of them vanishing, but not being singular, at p. Now by a Bertini approach, $n-1$ such hypersurfaces intersect transversely in B: the degree of the dual hypersurface is then

$$deg(((d-1)\lambda - e)^{n-1}(d\lambda - 2e)) = d(d-1)^{n-1} - 2.$$

□

Exercise 1.11. **??** Let $X \subset \mathbb{P}^n$ be a variety of degree d and dimension k; suppose that $p \in X$ is a point of multiplicity m (see Section **??** for the definition). Let $B = Bl_p(\mathbb{P}^n)$ be the blow-up of $\mathbb{P}^n$ at the point p, and $\tilde{X} \subset B$ the proper transform of X in B. Find the class $[\tilde{X}] \in A^{n-k}(B)$.

Solution to Exercise ??: The class of $\tilde{X}$ is going to be of the kind $\alpha\lambda^{n-k} + \beta e^{n-k}$. To get α, the intersection with a general $(n-k)$-plane disjoint from E will be d distinct and reduced points, so $\alpha = d$. To get β, let's intersect $\tilde{X}$ with a general $(n-k)$-plane inside E (whose class is $(-1)^{n-k-1}e^{n-k}$), and we get m distinct reduced points; this is because the intersection $\tilde{X} \cap E$ is a $(k-1)$-fold of degree m in $E \cong \mathbb{P}^{n-1}$. We get then

$$[\tilde{X}] = d\lambda^{n-k} + (-1)^{n-k-1}me^{n-k}.$$

□

Exercise 1.12. **??** Let $p \in X \subset \mathbb{P}^n$ be as in the preceding Exercise, and suppose that the projection map $\pi_p : X \to \mathbb{P}^{n-1}$ is birational onto its image. What is the degree of $\pi_p(X)$?

Solution to Exercise ??: The rational morphism π_p extends to a regular morphism $B \to \mathbb{P}^{n-1}$ (that we will call π), and we are looking for the degree of the image of $\tilde{X}$.

Let ζ be the hyperplane class in $\mathbb{P}^{n-1}$; we are looking for the intersection $\zeta^k \cdot [\pi(\tilde{X})]$; by the fact that π is birational on $\tilde{X}$, we have that this is the same as $\zeta^k \cdot \pi_*([\tilde{X}])$. By the push pull formula, this is equal to

$$deg(\pi_*(\pi^*(\zeta^k) \cdot [\tilde{X}])) = deg((\lambda - e)^k (d\lambda^{n-k} + (-1)^{n-k-1} m e^{n-k})) =$$

$$= deg(d\lambda^n + (-1)^{n-1} m e^n) = d - m,$$

where we used that ζ pulls back to the proper transform of an hyperplane containing p, so whose class is given by $\lambda - e$ by the previous exercise. □

Exercise 1.13. **??** Show that the Chow ring of a product of projective spaces $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}$ is

$$\begin{aligned} A(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}) &= \bigotimes A(\mathbb{P}^{r_i}) \\ &= \mathbb{Z}[\alpha_1, \ldots, \alpha_k]/(\alpha_1^{r_1+1}, \ldots, \alpha_k^{r_k+1}), \end{aligned}$$

where $\alpha_1, \ldots, \alpha_k$ are the pullbacks of the hyperplane classes from the factors. Use this to calculate the degree of the image of the Segre embedding

$$\sigma : \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k} \hookrightarrow \mathbb{P}^{(r_1+1)\cdots(r_k+1)-1}$$

corresponding to the multilinear map $V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$.

Solution to Exercise ??: The description of the Chow ring of $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_t}$ follows by exactly the same logic as in the two-factor case. Similarly, if $\zeta \in A^1(\mathbb{P}^{(r_1+1)\cdots(r_k+1)-1})$ is the hyperplane class in the target projective space, we see that $\sigma^*\zeta = \alpha_1 + \cdots + \alpha_k$, and the degree of the image is correspondingly

$$\deg\,(\alpha_1 + \cdots + \alpha_k)^{r_1+\cdots+r_k} = \binom{r_1 + \cdots + r_k}{r_1, \ldots, r_k} = \frac{(r_1 + \cdots + r_k)!}{r_1! \cdots r_k!}.$$

□

Exercise 1.14. **??** For $t \neq 0$, let $A_t : \mathbb{P}^r \to \mathbb{P}^r$ be the automorphism

$$[X_0, X_1, X_2, \ldots, X_r] \mapsto [X_0, tX_1, t^2X_2, \ldots, t^r X_r].$$

Describe the limit, as $t \to 0$, of the graph of A_t in $\mathbb{P}^r \times \mathbb{P}^r$: that is, let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^r \times \mathbb{P}^r$ be the closure of the locus

$$\tilde{\Phi} = \{(t, p, q) : t \neq 0 \text{ and } q = A_t(p)\}.$$

Describe the fiber Φ_0 of Φ over the point $t = 0$, and deduce once again the formula of Section **??** for the class of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$.

In the simplest case, this construction is a rational equivalence between a smooth plane section of a quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ (the diagonal, in terms of suitable identifications of the factors with $\mathbb{P}^1$), and a singular one (the sum of a line from each ruling), as in Figure **??**.

Solution to Exercise ??: In terms of coordinates $([X_0, \dots, X_r], [Y_0, \dots, Y_r])$ on $\mathbb{P}^r \times \mathbb{P}^r$, the limit is the locus

$$\Gamma = \bigcup_{i=0}^{r} V(X_0, \dots, X_{i-1}) \times V(Y_{i+1}, \dots, Y_r) \cong \bigcup_{i=0}^{r} \mathbb{P}^{r-i} \times \mathbb{P}^i,$$

whose class is visibly $\sum \alpha^i \beta^{r-i}$. To see the inclusion $\Phi_0 \subset \Gamma$, observe that the ideal of Φ includes the polynomials $X_i Y_j - t^{j-i} X_j Y_i$ for all $0 \le i < j \le r$; so the ideal of Φ_0 includes $X_i Y_j$ for all $i < j$. To see the inclusion in the opposite direction, show that

$$\big(0, [0, \dots, 0, 1, X_{i+1}, \dots, X_r], [Y_0, \dots, Y_{i-1}, 1, 0, \dots, 0]\big) = \lim_{t \to 0} \big(t, p, A_t(p)\big)$$

where

$$p = \big[t^i Y_0, t^{i-1} Y_1, \dots, t Y_{i-1}, 1, X_{i+1}, \dots, X_r\big].$$

□

Exercise 1.15. **??** Let

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

(Note that this includes all diagonals.) Show that this is a closed subvariety of codimension $n-1$ in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$.

Solution to Exercise ??: If the points p, q and r are given by the homogeneous vectors $(X_0, \dots, X_n)$, $(Y_0, \dots, Y_n)$ and $(Z_0, \dots, Z_n)$, then Ψ is the zero locus of the 3×3 minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \dots & X_n \\ Y_0 & Y_1 & \dots & Y_n \\ Z_0 & Z_1 & \dots & Z_n \end{pmatrix}.$$

These are homogeneous trilinear forms on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, from which we see that Ψ is indeed a closed subset of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. Moreover, from the form of the equations—the 3×3 minors of a $3 \times (n+1)$ matrix—we see that every component of Ψ has codimension at most $n-1$ (see for example Exercise 10.9 of **?**]). Since the projection $\Psi \to \mathbb{P}^n \times \mathbb{P}^n$ onto any two factors is surjective, with fibers $\mathbb{P}^1$ away from the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ and fibers $\mathbb{P}^n$ over Δ we can see that it's irreducible, and hence a subvariety of codimension $n-1$; This is because an eventual component lying over the diagonal would have codimension n, but this is not possible. □

Exercise 1.16. **??** Suppose that $(F_0, \dots, F_r)$ and $(G_0, \dots, G_r)$ are general $(r+1)$-tuples of homogeneous polynomials in $r+1$ variables, of degrees d and e respectively, so that in particular the maps $f : \mathbb{P}^r \to \mathbb{P}^r$ and $g : \mathbb{P}^r \to \mathbb{P}^r$ sending x to $(F_0(x), \dots, F_r(x))$ and $(G_0(x), \dots, G_r(x))$ are regular. For how many points $x = (x_0, \dots, x_r) \in \mathbb{P}^r$ do we have $f(x) = g(x)$?

Solution to Exercise ??: Let Γ_f and $\Gamma_g \subset \mathbb{P}^r \times \mathbb{P}^r$ be the graphs of f and g, and γ_f and $\gamma_g \in A^r(\mathbb{P}^r \times \mathbb{P}^r)$ the classes of their graphs. By Proposition **??**, we have

$$\gamma_f = \sum_{i=0}^{r} d^i \alpha^i \beta^{r-i} \quad \text{and} \quad \gamma_f = \sum_{i=0}^{r} e^i \alpha^i \beta^{r-i}$$

So, after verifying transversality either by Kleiman in characteristic 0 or more in general by calculation of tangent spaces, we have the answer

$$\#(\Gamma_f \cap \Gamma_g) = \deg(\gamma_f \cdot \gamma_g) = \sum_{i=0}^{r} d^i e^{r-i}.$$

□

Exercise 1.17. **??** Consider the locus $\Phi \subset (\mathbb{P}^2)^4$ of fourtuples of collinear points. Find the class $\varphi = [\Phi] \in A^2((\mathbb{P}^2)^4)$ of Φ by the method of undetermined coefficients; that is, by intersecting with cycles of complementary dimension.

Solution to Exercise ??: As suggested, we write

$$\varphi = [\Phi] = \sum_{i=1}^{4} c_i \alpha_i^2 + \sum_{1 \le i < j \le 4} d_{i,j} \alpha_i \alpha_j.$$

We then have

$$c_1 = \deg(\varphi \cdot \alpha_2^2 \alpha_3^2 \alpha_4^2) = \#\big(\Phi \cap (\mathbb{P}^2 \times \{p_0\} \times \{q_0\} \times \{r_0\})\big) = 0$$

for general (and in particular non-collinear) p_0, q_0 and $r_0 \in \mathbb{P}^2$; likewise, $c_i = 0$ for all i. After a transversality check, we have similarly

$$d_{1,2} = \deg(\varphi \cdot \alpha_1 \alpha_2 \alpha_3^2 \alpha_4^2) = \#\big(\Phi \cap (L \times M \times \{q_0\} \times \{r_0\})\big) = 1$$

for general lines $L, M \subset \mathbb{P}^2$ and points $q_0, r_0 \in \mathbb{P}^2$; likewise, $d_{i,j} = 1$ for all $i < j$. In sum, $\varphi = \sum_{i<j} \alpha_i \alpha_j$. □

Exercise 1.18. **??** With $\Phi \subset (\mathbb{P}^2)^4$ as in the preceding problem, calculate the class $\varphi = [\Phi]$ by using the result of Exercise **??** on the locus $\Psi \subset (\mathbb{P}^2)^3$ of triples of collinear points, and considering the intersection of the loci $\Psi_{1,2,3}$ and $\Psi_{1,2,4}$ of fourtuples (p_1, p_2, p_3, p_4) with p_1, p_2, p_3 collinear and with p_1, p_2, p_4 collinear.

Solution to Exercise ??: Let $\Psi_{i,j,k}$ be the locus where p_i, p_j and p_k are collinear, and $\psi_{i,j,k}$ its class. By Exercise **??** (or the calculation in Section **??**),

$$\psi_{1,2,3} = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{and} \quad \psi_{1,2,4} = \alpha_1 + \alpha_2 + \alpha_4$$

We observe that the loci $\Psi_{1,2,3}$ and $\Psi_{1,2,4}$ have intersection

$$\Psi_{1,2,3} \cap \Psi_{1,2,4} = \Phi \cup \Delta_{1,2}$$

where $\Delta_{1,2} = \{p_1 = p_2\}$ is the preimage of the diagonal under projection on the first two factors. Checking that this intersection is generically transverse (Kleiman won't help us here; we have to calculate tangent spaces), we have

$$\varphi + [\Delta_{1,2}] = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_4) = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2 + \sum_{i<j} \alpha_i\alpha_j$$

and the result follows. □

Exercise 1.19. **??** Let A, B and $C : \mathbb{P}^2 \to \mathbb{P}^2$ be three general automorphisms. For how many points $p \in \mathbb{P}^2$ are the points $p, A(p), B(p)$ and $C(p)$ collinear?

Solution to Exercise ??: In $(\mathbb{P}^2)^4$, let Γ_A, Γ_B and Γ_C be the pullbacks, via the projections $\pi_{1,2}, \pi_{1,3}$ and $\pi_{1,4}$, of the graphs of A, B and C; let γ_A, γ_B and γ_C be their classes. Kleiman's theorem implies that the intersection $\Phi \cap \Gamma_A \cap \Gamma_B \cap \Gamma_C$ is transverse, so that the answer to the exercise is given by

$$\begin{aligned}
\#(\Phi \cap \Gamma_A \cap \Gamma_B \cap \Gamma_C) &= \deg(\varphi\gamma_A\gamma_B\gamma_C) \\
&= \deg\left(\sum_{i<j} \alpha_i\alpha_j \prod_{k=2}^{4} (\alpha_1^2 + \alpha_1\alpha_k + \alpha_k^2)\right) \\
&= \deg\left(\alpha_2\alpha_3\alpha_4 \sum_{i<j} \alpha_i\alpha_j \prod_{k=2}^{4} (\alpha_1 + \alpha_k)\right) \\
&= 6
\end{aligned}$$

□

Exercise 1.20. **??** Let B be the blowup of $\mathbb{P}^n$ at a point p as in Section **??** above, with classes λ_k, γ_k and e_k as described. Use the relation $e_{n-1} = \lambda_{n-1} - \gamma_{n-1}$ to describe the classes γ_k in terms of λ_k and e_k and vice versa.

Solution to Exercise ??: A k-plane contained in the exceptional divisor E is the transverse intersection of E with the proper transform of a $(k+1)$-plane in $\mathbb{P}^n$ containing p; that is,

$$e_k = e\gamma_{k+1}.$$

Now write $e = \lambda - \gamma$ and use the relations derived in Section **??** to express this as

$$e_k = (\lambda - \gamma)\gamma_{k+1} = \lambda_k - \gamma_k.$$

Put another way, the difference between the class of the preimage of a k-plane not containing p and the class of the proper transform of a k-plane containing p is the class of a k-plane contained in E. Note that except in case $k = n - 1$ this is not directly visible! □

The next few exercises deal with the blow-up of $\mathbb{P}^3$ along a line. To fix notation, let $\pi : X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a line $L \subset \mathbb{P}^3$; that is, the graph $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ of the rational map $\pi_L : \mathbb{P}^3 \to \mathbb{P}^1$ given by projection from L; let $\alpha : X \to \mathbb{P}^1$ be projection on the second factor.

Exercise 1.21. **??** Let $H \subset \mathbb{P}^3$ be a plane containing L, and $\tilde{H} \subset X$ its proper transform. Let $J \subset \mathbb{P}^3$ be a plane transverse to L, $\tilde{J} \subset X$ its proper transform (which equals its preimage in X), and let $M \subset J$ be a line not meeting L. Show that the subvarieties

$$X, \quad \tilde{H}, \quad \tilde{J}, \quad \tilde{J} \cap \tilde{H}, \quad M, \quad M \cap \tilde{H}$$

are the closed strata of an affine stratification of X, with open strata isomorphic to affine spaces. In particular, since only one ($M \cap \tilde{H}$) is a point, deduce that $A^3(X) \cong \mathbb{Z}$.

Solution to Exercise ??: The solution of this exercise is left to the reader. □

Exercise 1.22. **??** Let $h = [\tilde{H}]$, $j = [\tilde{J}] \in A^1(X)$ and $m = [M] \in A^2(X)$ be the classes of the corresponding strata. Show that

$$h^2 = 0, \quad j^2 = m, \quad \text{and} \quad \deg(jm) = \deg(hm) = 1.$$

Conclude that

$$A(X) = \mathbb{Z}[h, j]/(h^2, j^3 - hj^2).$$

Solution to Exercise ??: As in the calculations in Section **??**, there are lots of effective cycles representing these classes, so it's not hard to evaluate the intersection products in question. For example, h is the class of the proper transform $\tilde{H}$ of a plane $H \subset \mathbb{P}^3$ containing L; if $H' \subset \mathbb{P}^3$ is another such plane, with proper transform $\tilde{H}'$, then $\tilde{H}$ and $\tilde{H}'$ will be disjoint, whence $h^2 = 0$. Similarly, two general planes not containing L will intersect transversely in a line disjoint from L, so $j^2 = m$, and so on. □

Exercise 1.23. **??** Now let $E \subset X$ be the exceptional divisor of the blow-up, and $e = [E] \in A^1(X)$ its class. What is the class e^2?

Solution to Exercise ??: In contrast to the last problem, the class e has E as its only effective representative cycle. To calculate e^2, we observe that the preimage in X of a plane $H \subset \mathbb{P}^3$ containing L is the sum of its proper transform and the exceptional divisor, so that $j = h + e$ or equivalently $e = j - h$. It follows then that

$$e^2 = (j - h)^2 = m - 2hj.$$

□

Exercise 1.24. **??** Let $\mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2$.

(a) Find the dimension and degree of the locus of double lines (in characteristic $\neq 2$).
(b) Find the dimension and degree of the locus Δ of singular conics (that is, line pairs and double lines).

Solution to Exercise ??: For the first part, observe that the locus of double lines is the image of the map $\mathbb{P}^2 \to \mathbb{P}^5$ sending a linear form to its square; in characteristic $\neq 2$, this is the Veronese embedding, so it has degree 5.

The second part can be done in many ways. We can do it in the manner of Section **??**: we realize the locus of singular conics as the image of $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ sending a pair of linear forms to their product. This is the Segre map $\sigma_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ followed by a linear projection. Since the projection has degree 2 onto its image, the degree of the image is $\frac{1}{2}\binom{4}{2} = 3$. Alternatively, if $L \subset \mathbb{P}^5$ is a general line—that is, a general pencil of conics—then L consists of all conics containing the base locus of the pencil, which consists of four points $p_1, \dots, p_4$, no three collinear. The points of $L \cap \Delta$ correspond to line pairs containing $\{p_1, \dots, p_4\}$, of which there are visibly 3. Finally, if we realize $\mathbb{P}^5$ as the space of 3×3 symmetric matrices, Δ is the zero locus of the determinant, which is a homogeneous cubic polynomial in the entries. □

Exercise 1.25. **??** Let $\mathbb{P}^9$ be the space of plane cubics, and $\Gamma \subset \mathbb{P}^9$ the locus of reducible cubics. Let L and $C \subset \mathbb{P}^2$ be a line and a smooth conic intersecting transversely at two points $p, q \in \mathbb{P}^2$; let $L + C$ be the corresponding point of Γ. Show that Γ is smooth at $L + C$, with tangent space

$$\mathbb{T}_{L+C}\Gamma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p) = F(q) = 0\}.$$

Solution to Exercise ??: Let $\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \Gamma \subset \mathbb{P}^9$ be as in Section **??**. Since (L, C) is the unique point in $\tau^{-1}(L + C)$, we need to show that the differential $d\tau$ is injective at (L, C), with image contained in the subspace specified. Letting G and H be the homogeneous linear and quadratic polynomials defining L and C, we can represent a tangent vector to $\mathbb{P}^2$ at L as $G + \epsilon G'$, where G' is a linear form taken modulo G, and likewise we can represent a tangent vector to $\mathbb{P}^5$ at C as $H + \epsilon H'$, where H' is a quadratic form taken modulo H. Multiplying, we have

$$d\tau_{(L,C)}(G + \epsilon G', H + \epsilon H') = GH + \epsilon(G'H + GH').$$

Thus the image of $d\tau_{(L,C)}$ contains all cubics of the form $G'H + GH'$, which is to say all cubics vanishing at $L \cap C$, and the result follows. □

Exercise 1.26. ?? Using the preceding exercise, show that if $p_1, \dots, p_7 \in \mathbb{P}^2$ are general points, and $H_i \subset \mathbb{P}^9$ is the hyperplane of cubics containing p_i, then the hyperplanes $H_1, \dots, H_7$ intersect Γ transversely—that is, the degree of Γ is the number of reducible cubics through $p_1, \dots, p_7$.

Solution to Exercise ??: Since the points p_i are general, no three are collinear and no six lie on a conic; thus if $L + C$ is any reducible cubic containing all 7, the line L must contain 2 and the conic C five. Say C contains $p_1, \dots, p_5$; since p_6 and p_7 are general with respect to $p_1, \dots, p_5$, L must be transverse to C and the points of $L \cap C$ disjoint from $p_1, \dots, p_7$.

Now let p_8 and p_9 be the points of intersection $L \cap C$. By the preceding exercise, we need to show that $L + C$ *is the unique cubic containing* $\{p_1, \dots, p_9\}$. But any such cubic contains the seven points $p_1, \dots, p_5, p_8, p_9$ of C and the four points $p_6, \dots, p_9$ of L, and so must contain both. □

Exercise 1.27. ?? Calculate the number of reducible plane cubics passing through 7 general points $p_1, \dots, p_7 \in \mathbb{P}^2$, and hence, by the preceding exercise, the degree of Γ.

Solution to Exercise ??: By the preceding exercises, this is equivalent to ask in how many ways can seven points be partitioned in two sets of two and five elements, that means, $\binom{7}{2} = 21$. □

Exercise 1.28. ?? We can also calculate the degree of the locus $\Sigma \subset \mathbb{P}^9$ of triangles (that is, totally reducible cubics) directly, as in the series of exercises starting with (**??**). To start, show that if $C = L_1 + L_2 + L_3$ is a triangle with the three vertices—that is, points $p_{i,j} = L_i \cap L_j$ of pairwise intersection—distinct, then Σ is smooth at C with tangent space

$$\mathbb{T}_{L+C}\Sigma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p_{i,j}) = 0 \ \forall\, i, j\}.$$

Solution to Exercise ??: We can represent the locus of triangles as the image of the map $\sigma : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$ sending a triple of linear forms to their product; we claim that at a point $(L_1, L_2, L_3) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ as in the exercise, the differential $d\sigma$ is injective, with image as specified. As in Exercise **??**, if $L_i = V(H_i)$ we write

$$\begin{aligned} d\sigma_{(L_1,L_2,L_3)}&(H_1 + \epsilon H_1', H_2 + \epsilon H_2', H_3 + \epsilon H_3') \\ &= H_1H_2H_3 + \epsilon(H_1'H_2H_3 + H_1H_2'H_3 + H_1H_2H_3'); \end{aligned}$$

and then it is sufficient to show that the pairwise products H_iH_j generate the ideal of all polynomials vanishing on the points $p_{i,j}$. □

Exercise 1.29. **??** Using the preceding exercise,

(a) Show that if $p_1, \dots, p_6 \in \mathbb{P}^2$ are general points, then the degree of Σ is the number of triangles containing $p_1, \dots, p_6$; and
(b) Calculate this number directly.

Solution to Exercise ??: The fact that for $p_1, \dots, p_6$ general the hyperplanes H_{p_i} intersect Σ transversely now follows as in the solution of Exercise **??**: any triangle containing $\{p_1, \dots, p_6\}$ must (after possibly re-ordering the points) consist of the union C of the lines $\overline{p_1, p_2}$, $\overline{p_3, p_4}$ and $\overline{p_5, p_6}$; we check that the points of pairwise intersection of these lines are distinct from each other and from $p_1, \dots, p_6$, and finally deduce that the only cubic containing these three points in addition to $p_1, \dots, p_6$ is C.

It follows that the degree of Σ is the number of triangles containing six general points, which is simply the number of ways of breaking up the points into three pairs; that is, $\frac{1}{6}\binom{6}{2,2,2} = 15$. □

Exercise 1.30. **??** Consider a general asterisk—that is, the sum $C = L_1 + L_2 + L_3$ of three distinct lines all passing through a point p. Show that the variety $\Sigma \subset \mathbb{P}^9$ of triangles is smooth at C, with tangent space the space of cubics double at p. Deduce that the space $A \subset \mathbb{P}^9$ of asterisks is also smooth at C.

Solution to Exercise ??: For the first part, this is the same calculation as in Exercise **??**, the difference being that here the quadrics $H_i H_j$ generate the ideal of polynomials double at p. For the second part, we observe that A is the image, under the restriction of the map $\sigma : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$, of the locus $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ of triples of concurrent lines, and argue (by homogeneity, for example) that Φ is smooth at (L_1, L_2, L_3). □

Exercise 1.31. **??** Let $p_1, \dots, p_5 \in \mathbb{P}^2$ be general points. Show that any asterisk containing $\{p_1, \dots, p_5\}$ consists, after possibly relabelling the points, of the sum of the line $L_1 = \overline{p_1, p_2}$, the line $L_2 = \overline{p_3, p_4}$ and the line $L_3 = \overline{p_5, (L_1 \cap L_2)}$.

Solution to Exercise ??: Since the points p_i are general, no line contains three of them; hence two must contain two and one one. □

Exercise 1.32. **??** Using the preceding two exercises, show that for $p_1, \dots, p_5 \in \mathbb{P}^2$ general points, the hyperplanes H_{p_i} intersect the locus $A \subset \mathbb{P}^9$ of asterisks transversely, and calculate the degree of A accordingly.

Solution to Exercise ??: This is slightly trickier than the preceding examples involving reducible cubics and triangles, since we don't have an explicit description of the tangent space $\mathbb{T}_C A \subset \mathbb{P}^9$. To set up, let $C = L_1 + L_2 + L_3$ be as in Exercise **??**, and suppose that $C' \in \mathbb{T}_C A \cap H_{p_1} \cap \cdots \cap H_{p_5}$. Since $\mathbb{T}_C A$ is contained in the locus of cubics double at p, the cubic C' must contain L_1 and L_2, so any tangent vector to A contained in the hyperplanes H_{p_i} must be of the form

$$d\sigma_{(L_1,L_2,L_3)}(H_1, H_2, H_3 + \epsilon H_3') = H_1 H_2 H_3 + \epsilon(H_1 H_2 H_3').$$

Since the tangent vector $(H_1, H_2, H_3 + \epsilon H_3')$ lies in $T_{(L_1,L_2,L_3)}\Phi$, the linear form H_3' must vanish at p and so must be a multiple of H_3; thus $C' = C$.

Thus the degree of A is simply the number of asterisks containing $p_1, \ldots, p_5$; by Exercise **??** this is $\frac{1}{2}\binom{5}{2,2,1} = 15$. □

Exercise 1.33. **??** Show that (in characteristic $\neq 3$) the locus $Z \subset \mathbb{P}^9$ of triple lines is a cubic Veronese surface, and deduce that its degree is 9.

Solution to Exercise ??: This is the same content as the remark right before Proposition **??**. □

Exercise 1.34. **??** Let $X \subset \mathbb{P}^9$ be the locus of cubics of the form $2L + M$ for L and M lines in $\mathbb{P}^2$.

(a) Show that X is the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under a regular map such that the pullback of a general hyperplane in $\mathbb{P}^9$ is a hypersurface of bidegree $(2, 1)$.
(b) Use this to find the degree of X.

Solution to Exercise ??: X is the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under the map τ sending a pair of linear forms (H, J) to the cubic form $H^2 J$, which is quadratic in the coefficients of H and linear in the coefficients of J; hence the first part.

Now, if $\alpha, \beta \in A^1(\mathbb{P}^2 \times \mathbb{P}^2)$ are the pullbacks of the hyperplane classes on $\mathbb{P}^2$ via the two projections, and $\zeta \in A^1(\mathbb{P}^9)$ the hyperplane class there, by the first part we have

$$\tau^*(\zeta) = 2\alpha + \beta.$$

Since the map τ is birational onto its image, it follows that the degree of $X \subset \mathbb{P}^9$ is given by

$$\deg X = \deg\left((2\alpha + \beta)^4\right) = 2^2 \binom{4}{2} = 24.$$

□

Exercise 1.35. **??** If you try to find the degree of the locus X of the preceding problem by intersecting X with hyperplanes $H_{p_1}, \ldots, H_{p_4}$, where

$$H_p = \{C \in \mathbb{P}^9 : p \in C\},$$

you get the wrong answer (according to the preceding problem). Why? Can you account for the discrepancy?

Solution to Exercise ??: First, the cardinality of the intersection $X \cap H_{p_1} \cap \cdots \cap H_{p_4}$ is easy to find: any nonreduced cubic containing all four points must be (after reordering) the cubic $2 \cdot \overline{p_1, p_2} + \overline{p_3 p_4}$. The number of such cubics is $\binom{4}{2} = 6$.

What's wrong? The problem is that, unlike the other examples of this technique, in this case *the hyperplanes H_{p_i} do not intersect the locus X transversely*. By the calculation analogous to those above, the tangent space to X at the image of (H, J) is the space of cubics in the ideal (HJ, H^2), which is contained in the hyperplanes H_{p_1} and H_{p_2}. □

Exercise 1.36. **??** Let $\mathbb{P}^2$ denote the space of lines in the plane, and $\mathbb{P}^5$ the space of plane conics. Let $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^5$ be the closure of the locus of pairs

$$\{(L, C) : \ C \text{ is smooth, and } L \text{ is tangent to } C\}.$$

Show that Φ is a hypersurface; and, assuming characteristic 0, find its class $[\Phi] \in A^1(\mathbb{P}^2 \times \mathbb{P}^5)$.

Solution to Exercise ??: To see that Φ is a hypersurface, let $F = \{(p, l) : p \in l\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$, and observe that Φ is the image of the incidence correspondence

$$\Omega = \{(C, L, p, l) : C \text{ and } L \text{ are tangent to } l \text{ at } p\} \subset \mathbb{P}^5 \times \mathbb{P}^{2*} \times F;$$

the dimension and irreducibility of Ω are readily seen via projection on the third factor.

As for the calculation of $[\Phi]$, this is probably best done by the method of undetermined coefficients: if α and $\beta \in A^1(\mathbb{P}^2 \times \mathbb{P}^5)$ are the pullbacks of hyperplane classes, then we can write

$$[\Phi] = c\alpha + d\beta$$

for some $c, d \in \mathbb{Z}$. If $\mathcal{C} \subset \mathbb{P}^2$ and $\mathcal{D} \subset \mathbb{P}^5$ are general pencils of lines and conics respectively, the integers c and d are then given as the cardinality of the intersection of Φ with the curves $\mathcal{C} \times \{C_0\}$ and $\{L_0\} \times \mathcal{D} \subset \mathbb{P}^2 \times \mathbb{P}^5$ respectively—that is, the number of lines in a general pencil that are tangent to a given conic, and the number of conics in a general pencil that a tangent to a given line. The answer in both cases is the number of branch points of a general degree 2 map from $\mathbb{P}^1$ to $\mathbb{P}^1$; that is, 2. Thus

$$[\Phi] = 2\alpha + 2\beta.$$

□

Exercise 1.37. **??** Now let $\mathbb{P}^9$ be the space of plane cubic curves as before, and let $Y \subset \mathbb{P}^9$ be the closure of the locus of reducible cubics consisting of a smooth conic and a tangent line. Use the result of the first part to determine the degree of B.

Solution to Exercise ??: Y is the image, under the map $\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9$ of Section **??**, of the locus $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^5$. Since this map is birational onto its image, and the pullback of the hyperplane class $\zeta \in A^1(\mathbb{P}^9)$ is given by $\tau^*(\zeta) = \alpha + \beta$, we have

$$\begin{aligned}\deg(Y) &= \deg\left(\tau^*(\zeta)^5[\Phi]\right)\\ &= \deg\left((\alpha+\beta)^5(2\alpha+2\beta)\right)\\ &= 42\end{aligned}$$

□

Exercise 1.38. ?? Let $\mathbb{P}^{14}$ be the space of quartic curves in $\mathbb{P}^2$, and let $\Sigma \subset \mathbb{P}^{14}$ be the closure of the space of reducible quartics. What are the irreducible components of Σ, and what are their dimensions and degrees?

Solution to Exercise ??: First, Σ has two irreducible components, one whose general point corresponds to a union of a line and a cubic, and one whose general point corresponds to a union of two conics. These are images of $\mathbb{P}^2 \times \mathbb{P}^9$ and $\mathbb{P}^5 \times \mathbb{P}^5$ respectively, under maps that are given by bihomogeneous forms of bidegree $(1, 1)$. The first map is birational onto its image, and so the first component has dimension 11 and degree $\binom{11}{3} = 1650$; the second map is finite of degree 2 onto its image, which thus has dimension 10 and degree $\frac{1}{2}\binom{10}{5} = 126$. □

Exercise 1.39. ?? Find the dimension and degree of the locus $\Omega \subset \mathbb{P}^{14}$ of totally reducible quartics (that is, quartic polynomials that factor as a product of four linear forms).

Solution to Exercise ??: Here Ω is the image of the map $\mu : (\mathbb{P}^2)^4 \to \mathbb{P}^{14}$ sending a fourtuple of linear forms to their product. Again, if $\alpha_i \in A^1\left((\mathbb{P}^2)^4\right)$ are the pullbacks of the hyperplane class from $\mathbb{P}^2$ via the four projections, and $\zeta \in A^1(\mathbb{P}^{14})$ is the hyperplane class there, then since the map μ is finite of degree $4! = 24$ we have

$$\begin{aligned}\deg(\Omega) &= \frac{1}{24}\deg\left(\tau^*(\zeta)^8\right)\\ &= \frac{1}{24}\deg\left((\alpha_1+\alpha_2+\alpha_3+\alpha_4)^8\right)\\ &= \frac{1}{24}\binom{8}{2,2,2,2}\\ &= 105\end{aligned}$$

□

Exercise 1.40. ?? Again let $\mathbb{P}^{14}$ be the space of plane quartic curves, and let $\Theta \subset \mathbb{P}^{14}$ be the locus of sums of four concurrent lines. Using the result of Exercise **??**, find the degree of Θ.

Solution to Exercise ??: Here Θ is the image, under the map μ of the preceding solution, of the locus $\Phi \subset (\mathbb{P}^2)^4$ described in Exercise **??**. Having calculated the class of Φ to be $\sum_{i<j} \alpha_i \alpha_j$ in that exercise, we have

$$\begin{aligned} \deg(\Theta) &= \frac{1}{24} \deg\left(\tau^*(\zeta)^6[\Phi]\right) \\ &= \frac{1}{24} \deg\left((\alpha_1+\alpha_2+\alpha_3+\alpha_4)^6 \sum_{i<j} \alpha_i \alpha_j\right) \\ &= \frac{1}{24} \cdot 6 \cdot \binom{6}{2,2,1,1} \\ &= 45 \end{aligned}$$

□

Exercise 1.41. **??** Find the degree of the locus $A \subset \mathbb{P}^{14}$ of the preceding problem, this time by calculating the number of sums of four concurrent lines containing six general points $p_1, \dots, p_6 \in \mathbb{P}^2$, assuming transversality.

Solution to Exercise ??: Given that no three of the points p_i are collinear, if a sum of four lines contains them all then either

(a) two of the lines will each contain two of the points, and the remaining two lines one each; or
(b) three of the lines will each contain two of the points.

If the four lines are concurrent (and the points general), the latter case can't occur; thus the sums of four collinear lines containing $p_1, \dots, p_6$ correspond to the decompositions of the set $\{p_1, \dots, p_6\}$ into two sets of two and two sets of 1 (take two of the lines to be the spans of the two pairs; take the remaining lines to be the lines spanned by the point of intersection of the first two and each of the two remaining points). The number is thus

$$\frac{1}{2^2}\binom{6}{2,2,1,1} = 45.$$

□

A natural generalization of the locus of asterisks, or of sums of four concurrent lines, would be the locus, in the space $\mathbb{P}^N$ of hypersurfaces of degree d in $\mathbb{P}^n$, of *cones*. We will indeed be able to calculate the degree of this locus in general, but it will require more advanced techniques than we have at our disposal here; see Section **??** for the answer.

Exercise 1.42. **??** Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d and $L \subset S$ a line.

Calculate the degree of the self-intersection of the class $\lambda = [L] \in A^1(S)$ by considering the intersection of S with a general plane $H \subset \mathbb{P}^3$ containing L.

Solution to Exercise ??: A general plane H containing L will intersect S in the union of L and a curve $C \subset H$ of degree $d-1$; applying Bertini, we can see that C intersects L transversely. If $\zeta \in A^1(S)$ is the restriction to S of the hyperplane class in $\mathbb{P}^3$, then we have

$$\lambda = \zeta - \gamma$$

where γ is the class of C; and

$$\deg(\lambda^2) = \deg\big(\lambda(\zeta - \gamma)\big) = 1 - (d-1) = 2 - d.$$

□

Exercise 1.43. **??** Let S be a smooth surface. Show that if $C \subset S$ is any irreducible curve such that the corresponding point in the Hilbert scheme $\mathcal{H}$ of curves on S lies on a positive-dimensioaln irreducible component of $\mathcal{H}$, then the degree $\deg(\gamma^2)$ of the self-intersection of the class $\gamma = [C] \in A^1(S)$ is nonnegative. Using this and the preceding exercise, prove the statement made in Section **??** that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines*.

Solution to Exercise ??: Let $B \subset \mathcal{H}$ be a curve, with the point $b \in B$ corresponding to C. If B is rational, we are essentially done: the universal family of curves over $B \subset \mathcal{H}$ gives a rational equivalence between C and the curve C' corresponding to any point $b' \in B$, and since C and C' have no common components their intersection number is nonnegative.

In general, whatever the genus of B, by Riemann-Roch a high multiple of the the point $b \in B$ will be rationally equivalent to a linear combination $\sum m_i b_i$ of other points of B; if $C_i \subset S$ is the curve corresponding to b_i we see similarly that $mC \sim \sum m_i C_i$; by the same token $\deg((m\gamma)^2)$, and hence $\deg(\gamma^2)$, will be nonnegative. (If B is singular we may work with its normalization.)

So, the Hilbert scheme of lines on a surface S of degree $d \geq 3$ is composed by isolated points (because the self intersection is negative) and they are finite because Hilbert schemes are proper. □

Exercise 1.44. **??** Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that

(a) if C has genus 2, it must lie on a quadric surface;
(b) if C has genus 1, it cannot lie on a smooth quadric surface (in fact, it can't lie on any quadric); and
(c) if C has genus 0, it may or may not lie on a quadric surface (that is, some rational quintic curves do lie on quadrics and some don't).

Solution to Exercise ??: For the first part, look at the restriction maps

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)).$$

$h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, and by Riemann-Roch, $h^0(\mathcal{O}_C(2)) = 10 - g + 1$. So if $g = 2$, the restriction map must have a nonzero kernel; that is, C lies on a quadric.

If $g = 1$, the fact that C can't lie on a smooth quadric follows from the genus formula for $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Finally, there do exist smooth rational quintics lying on a quadric: these are the curves of type $(1, 4)$. But a dimension count shows that the space of all rational quintic curves has dimension 20 (4-tuples of quintic polynomials form a 24-dimensional vector space; mod scalars and PGL_2, the space of image curves in $\mathbb{P}^3$ is 24-4=20-dimensional); while the dimension of the space of curves of type $(1, 4)$ on a quadric is $9+2\cdot5-1 = 18$ (for the quadric, which is uniquely determined by the curve, and the space of curves of type $(1, 4)$ on Q). □

Exercise 1.45. ?? Let $C \subset \mathbb{P}^3$ be a smooth quintic curve of genus 2. Show that C lies on a quadric surface Q and a cubic surface S with intersection $Q \cap S$ consisting of the union of C and a line.

Solution to Exercise ??: As in the last exercise, we start by looking at the restriction maps

$$r_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) \quad \text{and} \quad r_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3)).$$

As we found in Exercise **??**, the map r_2 must have a kernel, corresponding to a quadric Q containing C (unique, by Bezout, and necessarily irreducible). Similarly, the kernel of r_3 must have dimension at least 5; since the space of products LQ has vector space dimension 4, there must be a cubic surface S containing C and not containing Q. Q being irreducible, the intersection $Q \cap S$ will have dimension 1; and by Bezout it must consist of the union of C and a line. □

Exercise 1.46. ?? Use the result of Exercise **??**—showing that a smooth quintic curve of genus 2 is linked to a line in the complete intersection of a quadric and a cubic—to find the dimension of the subset of the Hilbert scheme corresponding to smooth curves of degree 5 and genus 2.

Solution to Exercise ??: Here we want to look at the incidence correspondence Φ of four-tuples (Q, S, C, L) consisting of a quadric surface Q, a cubic surface S, a smooth quintic curve of genus 2 and a line L such that

$$Q \cap S = C \cup L.$$

We can calculate the dimension of Φ via its projection to the last factor, and from that find that the space of smooth quintics of genus 2 has dimension 20. □

Exercise 1.47. **??** Let $X \subset \mathbb{P}^4$ be the affine cone defined by $xy - uv = 0$. Show that the conclusion of Theorem **??** fails.

Hint: on one hand, consider the intersection between the plane $\Gamma = V(x, u)$ and the line $L_1 = V(y, v, w)$; on the other hand, consider the intersection between Γ and the line $L_2 = (x, v, w)$. Then, show that the two lines are rationally equivalent, using transitive property of rational equivalence.

Solution to Exercise ??: Let Γ be the plane (x, u), and L_1 and L_2 the lines (y, v, w) and (x, v, w). It's easy to show that the intersection $\Gamma \cup L_1$ is empty, and the intersection $\Gamma \cup L_2$ is transverse and one point. Any intersection product on $A^*(X)$ satisfying Theorem **??** should then satisfy

$$deg([\Gamma] \cdot [L_1]) = 0 \quad \text{and} \quad deg([\Gamma] \cdot [L_1]) = 1$$

because both intersection are transverse. We will show now that the two lines L_1 and L_2 are indeed rationally equivalent in $A^2(X)$, so that the two conditions above lead to an absurd; the rational equivalence will be a composition of two. The first is inside the plane (y, v) from L_1 to the line (x, y, v); the second inside the plane (x, v) from the line (x, y, v) to L_2. Note that both planes are contained in X, so this actually gives a rational equivalence $L_1 \sim L_2$, and the claim follows. □

1.3 Chapter 3

Exercise 1.48. **??** Let Λ and $\Gamma \in G$ be two points in the Grassmannian $G = G(k, V)$. Show that the line $\overline{\Lambda, \Gamma} \subset \mathbb{P}(\wedge^k V)$ is contained in G if and only if the intersection $\Lambda \cap \Gamma \subset V$ of the corresponding subspaces of V has dimension $k - 1$.

Solution to Exercise ??: If $dim(\Lambda \cap \Gamma) = k - 1$, than we can choose a basis $v_1, \ldots, v_n$ of V such that in $\mathbb{P}(\wedge^k V)$

$$[\Lambda] = [v_1 \wedge \ldots \wedge v_{k-1} \wedge v_k] \quad \text{and} \quad [\Gamma] = [v_1 \wedge \ldots \wedge v_{k-1} \wedge v_{k+1}].$$

In this way, elements in the line joining $[\Lambda]$ and $[\Gamma]$ are given by tensors

$$[v_1 \wedge \ldots \wedge v_{k-1} \wedge (\alpha v_k + \beta v_{k+1})]$$

that are still pure tensors in $\mathbb{P}(\wedge^k V)$, so that the line is all contained in $G(k, V)$. Conversely, suppose $dim(\Lambda \cap \Gamma) = h < k - 1$, and let's again choose a basis of V such that

$$[\Lambda] = [v_1 \wedge \ldots \wedge v_h \wedge v_{h+1} \wedge \ldots \wedge v_k] \quad \text{and} \quad [\Gamma] = [v_1 \wedge \ldots \wedge v_h \wedge v_{k+1} \wedge \ldots \wedge v_{2k-h}].$$

Let's consider now a general element of the line $\overline{\Gamma, \Lambda}$, given by

$$[\eta] = [v_1 \wedge \ldots \wedge v_h \wedge (\alpha v_{h+1} \wedge \ldots \wedge v_k + \beta v_{k+1} \wedge \ldots \wedge v_{2k-h}].$$

It's easy to see that if both α and β are nonzero, the map $V \xrightarrow{\wedge\eta} \wedge^{k+1} V$ has rank $n - h \geq n - k$ (this can be seen explicitely checking the wedge products $\eta \wedge v_i$), so that this is not an element of $G(k, V)$. □

Exercise 1.49. **??** Using the fact that the Grassmannian

$$G = G(k, V) \subset \mathbb{P}(\wedge^k V)$$

is cut out by quadratic equations, show that if $[\Lambda] \in G$ is the point corresponding to a k-plane Λ then the tangent plane $\mathbb{T}_{[\Lambda]} G \subset \mathbb{P}(\wedge^k V)$ intersects G in the locus

$$G \cap \mathbb{T}_\Lambda G = \{\Gamma : \dim(\Gamma \cap \Lambda) \geq k - 1\};$$

that is, the locus of k-planes meeting Λ in codimension 1.

Solution to Exercise ??: It's easy to see that every line contained in G through $[\Lambda]$ is contained in $\mathbb{T}_\Lambda G$, and the union of lines through $[\Lambda]$ is exactly the locus of planes meeting Λ in dimension $k - 1$ from Exercise **??**, so we have the inclusion

$$G \cap \mathbb{T}_\Lambda G \supseteq \{\Gamma : \dim(\Gamma \cap \Lambda) \geq k - 1\}.$$

To see the other inclusion, let Γ be a point in $G \cap \mathbb{T}_\Lambda G$; this means that the line $\overline{\Gamma, \Lambda}$ intersects G in (at least) the two points Γ and Λ; this line is contained in the linear space $\mathbb{T}_\Lambda G$, so it's tangent to G at Λ; the intersection multiplicity of the line $\overline{\Gamma, \Lambda}$ with G is at least 3. Now, we know that G is the intersection of quadric hypersurfaces in $\mathbb{P}(\wedge^k V)$; the line $\overline{\Gamma, \Lambda}$ intersect with multiplicity at least 3 all these hypersurfaces; by Bezout's theorem then, this line is contained in all these hypersurfaces, so is contained in G; Γ belongs to a line in G through Λ, so the claim follows. □

Exercise 1.50. **??** Consider the universal k-plane over $G = \mathbb{G}(k, \mathbb{P}V)$:

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\},$$

whose fiber over a point $[\Lambda] \in G$ is the k-plane $\Lambda \subset \mathbb{P}V$. Show that this is a closed subvariety of $G \times \mathbb{P}V$ of dimension $k + (k + 1)(n - k)$, and that it's cut out on $G \times \mathbb{P}V$ by bilinear forms on $\mathbb{P}(\wedge^k V) \times \mathbb{P}V$.

Solution to Exercise ??: Let's consider the natural bilinear map $\wedge^k V \otimes V \to \wedge^{k+1} V$, and let's consider in $\wedge^k V \otimes V$ the inverse image of zero of this map; this leads to a subvariety of $\mathbb{P}(\wedge^k V) \times \mathbb{P}V$ cut out by bilinear forms. The fiber over a point $[\eta]$ of $\mathbb{P}(\wedge^k V)$ is given by all vectors $[v]$ such that $\eta \wedge v = 0$, so that over a point Λ of G we find exactly the vector space $\Lambda \subset \mathbb{P}V$. Restricting over G, then, we find the universal plane Φ as closed subvariety. Considering the projection onto G, we easily find that is irreducible, and of dimension $k + (k+1)(n-k)$. □

Exercise 1.51. **??** Use the preceding exercise to show that if $X \subset \mathbb{P}^n$ is any subvariety of dimension $l < n - k$, then the locus

$$\Gamma_X = \{\Lambda \in \mathbb{G}(k,n) \mid X \cap \Lambda \neq \varnothing\}$$

of k-planes meeting X is a closed subvariety of $\mathbb{G}(k,n)$ of codimension $n - k - l$.

Solution to Exercise ??: Using the previous exercise, let $\Phi \subset \mathbb{P}(\wedge^k V) \times \mathbb{P}^n$ be the universal plane. The inverse image of X from the second projection is the locus

$$\tilde{\Gamma}_X = \{(\Lambda, p) \mid p \in \Lambda, p \in X\}$$

so its projection in $\mathbb{P}(\wedge^k V)$ will be the locus Γ_X. By properness of the projection of the first projection, this locus is a closed subvariety. To find its dimension, note that the projection $\tilde{\Gamma}_X \to X$ is a fibration, whose fibers are all the k-planes through a given point, that is, $k(n-k)$ dimensional (this also proves that $\tilde{\Gamma}_X$ is irreducible); the dimension of $\tilde{\Gamma}_X$ is then $k(n-k) + l$; if we prove that the projection $\tilde{\Gamma}_X \to \Gamma_X$ is generically finite, then the claim about the dimension of Γ_X follows. Now, the general $n - l$-plane intersects X in finitely many points: inside these planes, we can easily find a k-plane that intersects X in finitely many points as well, that, together with the irreducibility of $\tilde{\Gamma}_X$, proves the assertion. □

Exercise 1.52. **??** Let $l < k < n$, and consider the locus of nested pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k:

$$\mathbb{F}(l,k;n) = \{(\Gamma, \Lambda) \in \mathbb{G}(l,n) \times \mathbb{G}(k,n) \mid \Gamma \subset \Lambda\}.$$

Show that this is a closed subvariety of $\mathbb{G}(l,n) \times \mathbb{G}(k,n)$, and calculate its dimension. (These are examples of a further generalization of Grassmannians called *flag manifolds*, which we'll explore further in Section **??**.)

Solution to Exercise ??: We will solve this exercise using a method that will be the main content in Chapter 5 (and more), getting $\mathbb{F}(l,k;n)$ as vanishing locus of a section of a vector bundle, that will automatically prove this is a closed subvariety. On $\mathbb{G}(l,n) \times \mathbb{G}(k,n)$, we have the following natural diagram of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^*\mathcal{S}_l & \xrightarrow{\iota_l} & \pi_1^*(\mathcal{O}_{\mathbb{G}(l,n)} \otimes V) & \xrightarrow{p_l} & \pi_1^*\mathcal{Q}_l & \longrightarrow & 0 \\ & & & & \downarrow \sim & & & & \\ 0 & \longrightarrow & \pi_2^*\mathcal{S}_k & \xrightarrow{\iota_k} & \pi_2^*(\mathcal{O}_{\mathbb{G}(k,n)} \otimes V) & \xrightarrow{p_k} & \pi_2^*\mathcal{Q}_k & \longrightarrow & 0 \end{array}$$

where the two middle vector bundles are canonically the same (trivial bundle); let's consider the composition

$$p_k \circ \iota_l : \pi_1^* \mathcal{S}_l \to \pi_2^* \mathcal{Q}_k.$$

On the fiber over a couple (Γ, Λ), this is the canonical map $\Gamma \to V/\Lambda$, and is zero exactly whether $\Gamma \subset \Lambda$, that means, over the flag variety. Seeing this map as a section of the vector bundle $(\pi_1^* \mathcal{S}_l)^* \otimes \pi_2^* \mathcal{Q}_k$, we get flag variety is the zero locus of a section of this bundle, so is a closed subvariety. To find the dimension, it's easy to get by the projection onto any of the two factors that the dimension is $(n-k)(l+1)-(l-k)^2$. □

Exercise 1.53. ?? Again let $l < k < n$, and for any $m \leq l$ consider the locus of pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k intersecting in dimension at least m:

$$\mathbb{F}(l, k; m; n) = \{(\Gamma, \Lambda) \in \mathbb{G}(l, n) \times \mathbb{G}(k, n) \mid \dim(\Gamma \cap \Lambda) \geq m\}.$$

Show that this is a closed subvariety of $\mathbb{G}(l, n) \times \mathbb{G}(k, n)$, and calculate its dimension.

Solution to Exercise ??: Consider the flag manifolds $\mathbb{F}(m, l; n)$ and $\mathbb{F}(m, k; n)$ as in the previous exercise. We can consider the intersection

$$\mathbb{F}(m, l; n) \times \mathbb{G}(k, n) \cap \mathbb{G}(l, n) \times \mathbb{F}(m, k; n) \subset \mathbb{G}(l, n) \times \mathbb{G}(m, n) \times \mathbb{G}(k, n)$$

that will consist of the locus

$$\Psi = \{(\Gamma, \Sigma, \Lambda) \mid \Sigma \subset \Gamma, \Sigma \subset \Lambda\}$$

and when we project it to $\mathbb{G}(l, n) \times \mathbb{G}(k, n)$ we get the desired locus as closed subvariety. By the projection onto the middle coordinate, it's easy to get that the dimension of Ψ is

$$(n-m)(m+1) + (n-l)(l-m) + (n-k)(k-m).$$

Then, the projection $\Psi \to \mathbb{F}(l, k; m; n)$ is generically one-to-one (because the locus where the fiber is not one point, that is, when the dimension of intersection is higher, is less dimensional), so that the dimension of $\mathbb{F}(l, k; m; n)$ is the same. □

Exercise 1.54. ?? Assume that the characteristic of our ground field is 0. Let $B \subset \mathbb{G}(1, n)$ be a curve in the Grassmannian of lines in $\mathbb{P}^n$, with the property that all nonzero tangent vectors to B have rank 1. Show that the lines in $\mathbb{P}^n$ parametrized by B either

(a) all lie in a fixed 2-plane;
(b) all pass through a fixed point; or
(c) are all tangent to a fixed curve $C \subset \mathbb{P}^n$.

(Note that the last possibility actually subsumes the first.)

Solution to Exercise ??: Let $\Phi \subset \mathbb{G}(1, n) \times \mathbb{P}^n$ be the universal line over $\mathbb{G}(1, n)$; at every point L of B, the tangent vector in $\mathbb{G}(1, n)$ has one dimensional kernel, so we can associate to it one point of L in the fiber of the universal line; we then get a map

$g : B \to \Phi$ obtained in this way (one should prove that this is actually an algebraic morphism though); projecting to the second factor $\mathbb{P}^n$, we get a map $f : B \to \mathbb{P}^n$, whose image we will call C. This curve C is composed by all points "around which" the line L locally rotates as we move along B (the rank 1 condition is exactly the condition of locally rotating around a point). If C is a point p, all lines corresponding to points of B contain p, so we are in situation (b); let's suppose now C is a curve: we will prove that for every point $[\Gamma] \in B$, we have $\mathbb{T}_{f(\Gamma)}C = L$, so that all points of B correspond to lines tangent to the curve C constructed in this way. Let now $(L, p) \in \Phi$, let's describe its tangent space inside

$$T_L\mathbb{G}(1,n) \times T_p\mathbb{P}^n = Hom(\tilde{L}, V/\tilde{L}) \times Hom(\tilde{p}, V/\tilde{p});$$

this is given by couples (α, β) such that the following diagram commutes

$$\begin{array}{ccc} \tilde{p} & \xrightarrow{\beta} & V/\tilde{p} \\ \iota\downarrow & & p\downarrow \\ \tilde{L} & \xrightarrow{\alpha} & V/\tilde{L} \end{array}$$

where vertical arrows are the natural ones. Now, we have that for points in $g(B)$, the composition $\alpha \circ \iota = p \circ \beta$ is zero, because p is chosen to be in the kernel of α; this means then that the image of β is the space $\tilde{L}/\tilde{p}$; but once we project onto $\mathbb{P}^n$, the tangent space to C is just given by β, and its image will determine the tangent line in $\mathbb{P}^n$; in this case then, the tangent line is exactly L, so the claim follows. □

Exercise 1.55. ?? Show that an automorphism of $G(k,n)$ carries tangent vectors to tangent vectors of the same rank (in the sense of Section **??**), and hence that in case $1 < k < n$ the group of automorphisms of $G(k,n)$ cannot act transitively on nonzero tangent vectors. Show, on the other hand, that the group of automorphisms of $G(k,n)$ *does* act transitively on tangent vectors of a given rank.

Solution to Exercise ??: The action of $PGL(V)$ on $G(k,n)$ extends to a linear action on the total space of the tangent bundle $TG(k,n)$. If $(\Gamma, v : \Gamma \to V/\Gamma)$ is an element of $TG(k,n)$, and $\varphi \in PGL(V)$, then

$$\varphi(\Gamma, v) = (\varphi(\Gamma), \varphi \circ v \circ \varphi^{-1})$$

where the last homomorphism is obtained by the following diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{v} & V/\Gamma \\ \varphi\downarrow & & \varphi\downarrow \\ \varphi(\Gamma) & \xrightarrow{\varphi \circ v \circ \varphi^{-1}} & V/\varphi(\Gamma) \end{array}$$

so that the rank of v can't change. Moreover, through this description, is an easy exercise in linear algebra proving that this action is indeed transitive on tangent vectors of given rank. □

Exercise 1.56. **??** In Example **??**, we showed that the open Schubert cell $\Sigma_1^\circ = \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{1,1})$ is isomorphic to affine space $\mathbb{A}^3$. For each of the remaining Schubert indices a, b, show that the Schubert cell $\Sigma_{a,b}^\circ \subset \mathbb{G}(1,3)$ is isomorphic to affine space of dimension $4 - a - b$.

Solution to Exercise ??: This exercise is left to the reader. □

Exercise 1.57. **??** Consider the Schubert cycle

$$\Sigma_1 = \{\Lambda \in \mathbb{G}(1,3) \mid \Lambda \cap L \neq \varnothing\}.$$

Suppose $\Lambda \in \Sigma_1$ and that $\Lambda \neq L$, so that $\Lambda \cap L$ is a point q and the span $\overline{\Lambda, L}$ a plane K. Show that Λ is a smooth point of Σ_1, and that its tangent space is

$$T_\Lambda(\Sigma_1) = \{\varphi \in \operatorname{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) \mid \varphi(\tilde{q}) \subset \tilde{K}/\tilde{\Lambda}\}.$$

Solution to Exercise ??: This exercise and the following are particular cases of Theorem **??** in next chapter. As in there, we will solve this exercise choosing a suitable coordinate system. Let x_0, x_1, x_2, x_3 be coordinates such that $\Lambda = V(x_2, x_3)$ and $L = V(x_1, x_3)$. Let's pick $\Gamma = V(x_0, x_1)$ and let's work in the affine chart U_Γ. Here, we have coordinates a, b, c, d such that the point (a, b, c, d) corresponds to the span of $[1, 0, a, b]$ and $[0, 1, c, d]$. As in Proposition **??**, we can identify U_Γ with $T_\Lambda\mathbb{G}(1,3)$ and under this correspondence, the point (a, b, c, d) correspond to the morphism

$$\begin{aligned} \Gamma &\xrightarrow{\ \varphi\ } V/\Gamma \\ (1,0,0,0) &\longrightarrow (0,0,a,b) + \Gamma \\ (0,1,0,0) &\longrightarrow (0,0,c,d) + \Gamma. \end{aligned}$$

Now, we can characterize in these coordinates the condition of meeting L; the condition is just for the four points $[1, 0, a, b]$, $[0, 1, c, d]$, $[1, 0, 0, 0]$ and $[0, 0, 1, 0]$ (the two latter spanning L) to be coplanar; the condition is then of the determinant of the resulting matrix being zero, that is, $b = 0$. Reading this condition in $T_\Lambda\mathbb{G}(1,3)$, this is the condition of $(1, 0, 0, 0)$ to land in $V(x_4)$, that is, $\varphi(\tilde{q}) \subseteq \tilde{K}/\tilde{\Lambda}$. □

Exercise 1.58. **??** Consider the Schubert cycle

$$\Sigma_{2,1} = \Sigma_{2,1}(p, H) = \{\Lambda \in \mathbb{G}(1,3) \mid p \in \Lambda \subset H\}.$$

Show that $\Sigma_{2,1}$ is smooth, and that its tangent space at a point Λ is

$$T_\Lambda(\Sigma_{2.1}) = \{\varphi \in \operatorname{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) \mid \varphi(\tilde{p}) = 0 \text{ and } \operatorname{Im}(\varphi) \subset \tilde{H}/\tilde{\Lambda}\}.$$

Solution to Exercise ??: As the previous exercise, this is a particular case of Theorem **??**. In this case though, we can work more directly, as in Proposition **??**; this is because in both cases the Schubert cycles are linear spaces in the Plücker embedding. As in Proposition **??**, let's work in an open set U_Γ, that we can identify with $T_\Lambda \mathbb{G}(1,3)$. Inside $V \cong \tilde{\Lambda} \oplus \Gamma$, we have the linear spaces that are still in $\Sigma_{2,1}$, as the ones still containing the line $\tilde{p}$, and still contained in the space $\tilde{H}$; looking at them as graphs of linear functions $\tilde{\Lambda} \to \Gamma \cong V/\Lambda$, we get the two desired conditions $\varphi(\tilde{p}) = 0$ and $\varphi(\tilde{\Lambda}) \subset \tilde{K}/\tilde{\Lambda}$, from which smoothness follows by dimensional considerations. □

Exercise 1.59. **??** Use the preceding two exercises to show in arbitrary characteristic that general Schubert cycles Σ_1 and $\Sigma_{2,1} \subset \mathbb{G}(1,3)$ intersect transversely, and deduce the equality $\deg(\sigma_1 \cdot \sigma_{2,1}) = 1$.

Solution to Exercise ??: Let Λ be the intersection of two general Schubert cycles Σ_1 and $\Sigma_{2,1}$. On tangent spaces, we have

$$T_\Lambda(\Sigma_1) = \{\varphi \in \operatorname{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) \mid \varphi(\tilde{q}) \subset \tilde{K}/\tilde{\Lambda}\}$$

$$T_\Lambda(\Sigma_{2.1}) = \{\varphi \in \operatorname{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) \mid \varphi(\tilde{p}) = 0 \text{ and } \operatorname{Im}(\varphi) \subset \tilde{H}/\tilde{\Lambda}\}$$

for general p, q, K, H; we can choose then p and q to generate Λ, and H and K to be independent in $V/\tilde{\Lambda}$. Gluing the two conditions, that is, intersecting the tangent spaces, we get

$$\varphi(\tilde{p}) = 0 \text{ and } \varphi(\tilde{q}) \subseteq \tilde{K}/\tilde{\Lambda} \cap \tilde{H}/\tilde{\Lambda} = 0$$

so the cycles are indeed transverse, and together with Theorem **??** the claim follows in every characteristic. □

Exercise 1.60. **??** Let $L_1, \ldots, L_4 \subset \mathbb{P}^3$ be four pairwise skew lines, and $\Lambda \subset \mathbb{P}^3$ a line meeting all four; set

$$p_i = \Lambda \cap L_i \quad \text{and} \quad H_i = \overline{\Lambda, L_i}.$$

Show that $[\Lambda] \in G$ fails to be a transverse point of intersection of the Schubert cycles $\Sigma_1(L_i)$ exactly when the cross-ratio of the four points $p_1, \ldots, p_4 \in \Lambda$ equals the cross-ratio of the four planes $H_1, \ldots, H_4$ in the pencil of planes containing Λ.

Solution to Exercise ??: Looking at tangent spaces, we have

$$T_\Lambda \Sigma_1(L_i) = \{\varphi \in \operatorname{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) \mid \varphi(\tilde{p}_i) \subset \tilde{H}_i/\tilde{\Lambda}\}$$

Having a nonzero element in the intersection correspond to have a morphism $\Lambda \to \mathbb{P}(V\tilde{\Lambda})$ sending p_i to $[H_i]$ for every i. Such a morphism exists if and only if the two cross ratios are the same, and this of course does not happen if the lines are general. □

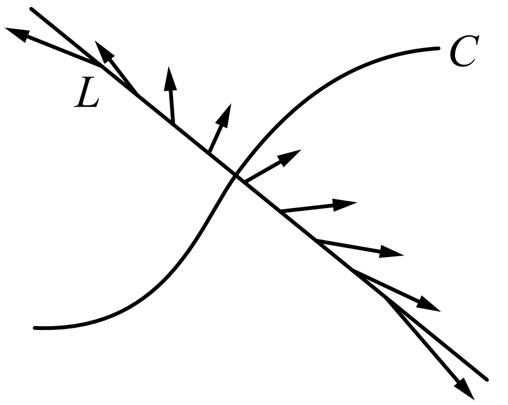


Figure 1.1 Deformation of a line L preserving contact with a curve C.
??

Exercise 1.61. **??** Let $C \subset \mathbb{P}^3$ be any curve, and $L \subset \mathbb{P}^3$ a line meeting C at one smooth point p of C and not tangent to C. Show that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting C is smooth at the point $[L]$, and that its tangent space at $[L]$ is the space of linear maps $\tilde{L} \to K^4/\tilde{L}$ carrying the one-dimensional subspace $\tilde{p} \subset \tilde{L}$ to the one-dimensional subspace $(\tilde{\mathbb{T}}_p C + \tilde{L})/\tilde{L}$ of $K^4/\tilde{L}$ (see Figure **??**).

Solution to Exercise ??: As in Exercise **??**, let's choose coordinates x_0, x_1, x_2, x_3 and a line Γ such that $\Lambda = V(x_2, x_3)$, $\mathbb{T}_p C = V(x_1, x_3)$, $\Gamma = V(x_0, x_1)$ and both U_Γ and $T_\Lambda \mathbb{G}(1,3)$ have coordinates (a, b, c, d); as above, the point (a, b, c, d) will correspond to the line

$$V(x_2 - ax_0 - cx_1, x_3 - bx_0 - dx_1).$$

Let's express C, locally around q, as a complete intersection of two homogeneous polynomials r and s in $\mathbb{P}^3$ of degrees d and e respectively; In our coordinate system, these two polynomials have can't include either monomials x_0^d and x_0^e (so that they contain q) or monomials $x_0^{d-1}x_2$ and $x_0^{e-1}x_2$ (so that their intersection have tangent space $\mathbb{T}_p C$ at p). The condition on a, b, c, d to meet C translates into the variety

$$V(r, s, x_2 - ax_0 - cx_1, x_3 - bx_0 - dx_1)$$

being nonempty; this means, the two homogeneous polynomials

$$\tilde{r}(x_0, x_1) = r(x_0, x_1, ax_0 + cx_1, bx_0 + dx_1)$$

$$\tilde{s}(x_0, x_1) = s(x_0, x_1, ax_0 + cx_1, bx_0 + dx_1)$$

having a common root. We can use then the theory of resultants, that will be treated in Chapter **??**; calling $r_0, \ldots, r_d, s_0, \ldots, s_e$ the coefficients of the polynomials $\tilde{r}$ and $\tilde{s}$ above (note that these coefficients are indeed polynomials in a, b, c, d), the condition on

a, b, c, d for $\tilde{r}$ and $\tilde{s}$ to have a common root is the vanishing of the determinant of the following matrix:

$$\begin{pmatrix}
r_0 & r_1 & \dots & \dots & r_d & 0 & 0 & \dots & \dots & 0 \\
0 & r_0 & r_1 & \dots & r_{d-1} & r_d & 0 & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
0 & 0 & \dots & \dots & r_0 & r_1 & \dots & \dots & \dots & r_d \\
s_0 & s_1 & \dots & \dots & s_{e-1} & s_e & 0 & 0 & \dots & 0 \\
0 & s_0 & s_1 & \dots & s_e & s_{e-1} & s_e & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
0 & 0 & \dots & \dots & \dots & s_{e-d+1} & \dots & \dots & & \dots s_e
\end{pmatrix}.$$

We are interested only in the linear parts of this equation in a, b, c, d, that will give us the condition on the tangent space; it's easy to show, from the conditions on r and s, that r_0 and s_0 have linear part consisting on a scalar multiple of b; further calculations, involving the fact that the intersection $V(r) \cap V(s) \cap \Gamma$ is tranverse, show that the linear part of this determinant is indeed just a scalar multiple of b; the condition on the tangent space is then $b = 0$, and the claim follows as in Exercise **??**. □

Exercise 1.62. **??** Let $B_1, \dots, B_4 \subset \mathbb{P}^3$ be four irreducible curves, and let $\varphi_1, \dots, \varphi_4 \in PGL_4$ be four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the incidence correspondence

$$\Phi = \{(\varphi_1, \dots, \varphi_4, L) \in (PGL_4)^4 \times \mathbb{G}(1,3) \mid L \cap \varphi_i(B_i) \neq \varnothing \ \forall i\}$$

is irreducible.

Solution to Exercise ??: First, a preliminary lemma: given p, q points of $\mathbb{P}^3$, the subvariety of PGL_4 of automorphism sending p to q is irreducible; if $p = q$, then this is the subgroup $Fix(p)$ of dimension 12, so it's irreducible; if $p \neq q$, this is the subvariety $\varphi \cdot \mathrm{Fix}(p)$, for φ any automorphism sending p to q, so is a translate of $Fix(p)$, so it's irreducible as well (this lemma can be generalized in many directions), of dimension 12. Let's consider now the bigger incidence correspondence

$$\Psi = \{(\varphi_1, \dots, \varphi_4, p_1, \dots, p_4, q_1, \dots, q_4, L) \in$$

$$\in (PGL_4)^4 \times (\mathbb{P}^3)^4 \times (\mathbb{P}^3)^4 \times \mathbb{G}(1,3) \mid \ p_i \in B_i, \ \varphi_i(p_i) = q_i, \ q_i \in L\}$$

that projects onto Φ via the projection onto the first and the fourth factor $\pi_{1,4}$. Let's project now onto the last three factors

$$(\mathbb{P}^3)^4 \times (\mathbb{P}^3)^4 \times \mathbb{G}(1,3);$$

the fiber over every point is the subvariety of $(PGL_4)^4$ sending p_i to q_i for $i = 1, 2, 3, 4$, irreducible of dimension 48 because of the preliminary lemma. Projecting down again onto the last two factors

$$(\mathbb{P}^3)^4 \times \mathbb{G}(1,3),$$

the fibers now are just the product of the four curves B_i, so fibers are irreducible again of dimension 4 by hypothesis. Projecting down again onto the last factor

$$\mathbb{G}(1,3),$$

fibers are now 4-tuples of points on lines, so a product of four $\mathbb{P}^1$, so irreducible again; the image is the whole $\mathbb{G}(1,3)$, irreducible. Concluding, we built up Ψ from three successive fibrations with irreducible fibers, with an irreducible base: Ψ is then irreducible, of dimension 60, so as is image by π_{14}, that is, Φ. □

Exercise 1.63. **??** Let $B_1, \dots, B_4 \subset \mathbb{P}^3$ be four curves, and $\varphi_1, \dots, \varphi_4 \subset PGL_4$ four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the set of lines $L \subset \mathbb{P}^3$ meeting C_1, C_2, C_3 and C_4 is finite; and that for any such L

(a) L meets each C_i at only one point p_i;
(b) p_i is a smooth point of C_i; and
(c) L is not tangent to C_i for any i.

Solution to Exercise ??: Let's keep the notation of Exercise **??**; we have that Ψ is 60-dimensional and irreducible, and projects onto $(PGL_4)^4$ that is 60-dimensional as well. So, either the map is generally finite of a given degree, or it has positive dimensional fibers and is not surjective; there are many ways of showing that the latter can't actually happen; one is appealing to principle (a) in Section 1.1.2: fibers are the set (actually, surjects onto)

$$\Gamma_{\varphi_1(B_1)} \cap \Gamma_{\varphi_2(B_2)} \cap \Gamma_{\varphi_3(B_3)} \cap \Gamma_{\varphi_4(B_4)}$$

whose intersection number is positive; hence, this intersection is nonempty for every element of $(PGL_4)^4$, so the projection is indeed surjective and the general fiber is finite. Let's prove now the second statement; to prove (a), consider the following incidence correspondence:

$$\Psi_{j,secant} = \big\{(\varphi_1, \dots, \varphi_4, p_1, \dots, p_4, \tilde{p}_j, q_1, \dots, q_4, \tilde{q}_j, L) \in$$

$$\in (PGL_4)^4 \times (\mathbb{P}^3)^5 \times (\mathbb{P}^3)^5 \times \mathbb{G}(1,3) \mid$$

$$\mid \; p_i \in B_i, \; \tilde{p}_j \in B_j, \; \tilde{p}_j \neq p_j, \; \varphi_i(p_i) = q_i, \; \varphi_j(\tilde{p}_j) = \tilde{q}_j, \; q_i \in L, \; \tilde{q}_j \in L\big\}$$

of situations in which L meets the curve $\varphi_j(B_j)$ also anothe point $\tilde{q}_j$; a calculation as in the previous exercise shows that $\Psi_{j,secant}$ is indeed 59-dimensional; its projection onto $(PGL_4)^4$ won't be dominant, so the general fiber will avoid situation (a) for $j = 1, 2, 3, 4$. To prove (b) and (c), we consider the incidence correspondences

$$\Psi_{j,sing} = \big\{(\varphi_1, \dots, \varphi_4, p_1, \dots, p_4, q_1, \dots, q_4, L) \in$$

$$\in (PGL_4)^4 \times (\mathbb{P}^3)^4 \times (\mathbb{P}^3)^4 \times \mathbb{G}(1,3) \mid$$

$$\mid \ p_i \in B_i,\ p_j \in Sing(B_j),\ \varphi_i(p_i) = q_i,\ q_i \in L\big\}$$

$$\Psi_{j,tang} = \big\{(\varphi_1, \dots, \varphi_4, p_1, \dots, p_4, q_1, \dots, q_4, L) \in$$

$$\in (PGL_4)^4 \times (\mathbb{P}^3)^4 \times (\mathbb{P}^3)^4 \times \mathbb{G}(1,3) \mid$$

$$\mid \ p_i \in B_i,\ p_j \notin Sing(B_j),\ \varphi_i(p_i) = q_i,\ q_i \in L,\ L = \mathbb{T}_{q_j} C_j\big\}$$

that again is easy to prove are respectively 59 and 58 dimensional. The claim then follows as for (a). □

Exercise 1.64. **??** Let $C_1, \dots, C_4 \subset \mathbb{P}^3$ be any four curves, and $L \subset \mathbb{P}^3$ a line meeting all four and satisfying the conclusions of Exercise **??**. Use the result of Exercise **??** to give a necessary and sufficient condition that the four cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ intersect transversely at $[L]$, and show directly that this condition is satisfied when the C_i are general translates of given curves.

Solution to Exercise ??: From the hypothesis, we can apply Exercise **??**; calling $q_i = L \cap C_i$, we have

$$T_{[L]}\Gamma_{C_i} = \{\alpha : \tilde{L} \to V/\tilde{L} \mid \alpha(\tilde{q}_i) \subseteq (\tilde{\mathbb{T}}_{q_i} C_i + \tilde{L})/\tilde{L}\}.$$

The condition of the intersection of these four being 0 follows from Exercise **??**; to be more precise, if the four lines $\mathbb{T}_{q_i} C_i$ are skew, then the condition is of the two cross ratios to be different. If the four lines are not skew, one can check case by case what situations are admissible and which are not. Let's go back to the previous exercise then, and show that if $\varphi_1, \dots, \varphi_4$ are general, we have the four points q_i different, and the four planes $\mathbb{T}_{q_i} C_i + L$ different (these two conditions together assure the four lines being skew) and the cross ratios different too. From our analysis of the incidence correspondence Ψ in Exercise **??**, it's easy to see that the the subset for which the q_is are different each other is open. About the four planes, we have to consider the following incidence correspondence

$$\Psi_{j_1,j_2,coplanar} = \big\{(\varphi_1, \dots, \varphi_4, p_1, \dots, p_4, H, q_1, \dots, q_4, L) \in$$

$$\in (PGL_4)^4 \times (\mathbb{P}^3)^4 \times \mathbb{P}^{3*} \times (\mathbb{P}^3)^4 \times \mathbb{G}(1,3) \mid$$

$$| \ p_i \in B_i,\ \varphi_i(p_i) = q_i,\ q_i \in L \subset H,\ \mathbb{T}_{q_{j_1}} C_{j_1} \subset H,\ \mathbb{T}_{q_{j_2}} C_{j_2} \subset H \big\}$$

that by arguments as in previous exercises is 59-dimensional, so for general translates, the tangent lines are actually skew. Finally, consider the incidence correspondence

$$\Psi_f = \big\{(\varphi_1, \ldots, \varphi_4, q_1, \ldots, q_4, H_1, \ldots, H_4, L) \in$$

$$\in (PGL_4)^4 \times (\mathbb{P}^3)^4 \times (\mathbb{P}^{3*})^4 \times \mathbb{G}(1,3) \ |$$

$$| \ q_i = L \cap \varphi_i(B_i),\ H_i = L + \mathbb{T}_{q_i} \varphi_i(B_i),\ \text{conditions (a), (b), (c) hold} \big\}$$

where conditions (a), (b), (c) refer to Exercise **??**. This projects dominantly onto $(PGL_4)^4$ again; let's look at the map $CR : \Psi_f \to \mathbb{P}^1 \times \mathbb{P}^1$ given by the two croos ratios of the q_is and the H_is; we are interested in the inverse image of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, that is, situations in which the two cross ratios are different. If we show that CR is dominant, we are done, because then having different cross ratios will be an open condition. But this is easy to show: once we project onto

$$(\mathbb{P}^3)^4 \times (\mathbb{P}^{3*})^4 \times \mathbb{G}(1,3),$$

we get all possible 4-tuples of points and 4-tuples of planes containing th same line, so all cross ratios will appear. □

Exercise 1.65. **??** Let $C \subset \mathbb{P}^3$ be a smooth curve, and $p \in \mathbb{P}^3$ a general point. Show that

(a) p does not lie on any tangent line to C;
(b) p does not lie on any trisecant line to C; and
(c) p does not lie on any *stationary secant* to C; that is, a secant line $\overline{q,r}$ to C such that the tangent lines $\mathbb{T}_q C \cap \mathbb{T}_r C \neq \varnothing$.

Deduce from these facts that the projection $\pi_p : C \to \mathbb{P}^2$ is birational onto a plane curve $C_0 \subset \mathbb{P}^2$ having only nodes as singularities.

Solution to Exercise ??: It's easy to see that points lying on a tangent line to C are a subset of dimension 2 of $\mathbb{P}^3$, so a general point doesn't lie on it; if one want to make things precise, the incidence correspondence

$$\Phi_{tang} = \{(p,q) \in \mathbb{P}^3 \times C \ | \ p \in \mathbb{T}_q C\}$$

has one dimensional fibers over C, so it has dimension 2 and can't dominate $\mathbb{P}^3$.

About point (b), suppose trisecant lines cover an open subset of $\mathbb{P}^3$; then, the locus of trisecant lines to C in $\mathbb{G}(1,3)$ should be at least two dimensional; in fact, any curve in $\mathbb{G}(1,3)$ sweep out only a surface in $\mathbb{P}^3$, for obvious reasons. Now, locus of bisecant lines to a smooth curve in $\mathbb{P}^3$ is a surface in $\mathbb{G}(1,3)$, with a map from $C^{(2)}$ into it, so irreducible; if the locus of trisecant lines is two dimensional, it must be the same as the locus of bisecant lines; in other words, all bisecant lines to C are indeed trisecant. Let's now take a general hyperplane section of C; this consists of d points in a plane, and the above condition means that every line containing two of them does contain three of them. But for a general hyperplane section of a smooth curve, we have the *general position lemma* **??** (also in **?**], III.1, pag. 109), that says that the points are in general position; thus, no three of them can be collinear; this is an absurd.

About point (c), as for point (b), for stationary secants to cover an open subset of $\mathbb{P}^3$, all secants have to be stationary, that means, every two tangent lines to C intersect. Now, it is easy to show that any subset $A \subset \mathbb{G}(1,3)$ of pairwise intersecting lines is either contained in the set of lines in a given plane, or through a given point. In the first case the curve itself is planar, so a general point will not lie in any secant to it; the second case is impossible for smooth curves unless C is a line (so, planar again).

To conclude, the projection map will be injective on tangent spaces by point (a), at most 2 to 1 by point (b), and in case of two points mapping into one, it consists of two transverse branches by point (c); the map is then the (birational) map onto a nodal plane curve. □

Exercises **??**-**??** deal with the approach, described in Section **??**, to calculating the class of the variety $\Sigma_C \subset \mathbb{G}(1,3)$ of lines incident to a space curve $C \subset \mathbb{P}^3$ by specialization. Recall from that section that we choose a general plane $H \subset \mathbb{P}^3$ meeting C at d points p_i and a general point $q \in \mathbb{P}^3$, and let $\{A_t\}$ be the one-parameter subgroup of PGL_4 with attractor q and repellor H; we let $C_t = A_t(C)$ and take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\Psi^\circ = \{(t,\Lambda) \mid t \neq 0 \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

Exercise 1.66. **??** Show that the support of the fiber Ψ_0 is exactly the union of the Schubert cycles $\Sigma_1(\overline{p_i, q})$.

Solution to Exercise ??: At first, by Exercise **??**, the limit C_0 is the union of the lines $\overline{p_i, q}$. Let now $f : \mathbb{A}^1_t \to \mathbb{G}(1,3)$ be a curve in the Grassmannian such that its graph is contained in Ψ; in other words, such that $f(t)$ intersect C_t for $t \neq 0$; it is easy to show now, for instance using an incidence correspondence, that in this situation $f(0)$ does intersect C_0, that is, the union of the lines $\overline{p_i, q}$; this tells us that the support of the fiber Ψ_0 is contained in the union of the cycles $\Sigma_1(\overline{p_i, q})$. To prove the other inclusion, let L a general line in $\Sigma_1(\overline{p_{i_0}, q})$; so, the line will intersect $\overline{p_{i_0}, q}$ in a point r; let's pick also a general other point s on L, away from all curves C_t. Let's show that L is indeed in the limit Ψ_0, that is, can be obtained as limit of lines in Ψ_t; we know that r is in the limit of the curves C_t, so we have an arc $r(t)$ such that $r(t) \in C_t$ for $t \neq 0$ and $r(0) = r$. If we consider the family of lines $L_t = \overline{r(t), s}$, we have that $L_t \in \Psi_t$ for $t \neq 0$, and $L_t \to L$; the line L is then in the limit, and the claim follows. □

Exercise 1.67. **??** Show that Ψ_0 has multiplicity 1 at a general point of each Schubert cycle $\Sigma_1(\overline{p_i, q})$.

Solution to Exercise ??: We will apply the following criterion: a component Z in a flat limit $X_t \to X_0$ inside $X \subset \mathbb{A}^1 \times \mathbb{P}^n$ appear with multiplicity 1 if and only if there exist a curve $f : \mathbb{A}^1 \to X$ such that $f(t) \in X_t$ and $f(0) \in Z$; this is easy to show, because in such a case the intersection $X \cap \{0\} \times \mathbb{P}^n$ is transverse. Note that we already used this criterion in the previous exercise, when we claimed the existence of the arc $r(t)$, as consequence of the fact that the line $\overline{p_{i_0}, q}$ has multiplicity 1 as limit of the curves C_t. But now, the family of lines L_t built in the previous exercise satisfies the criterion for the family Ψ in the central component $\Sigma_1(\overline{p_{i_0}, q})$; the multiplicity is then 1. □

Exercise 1.68. **??** Suppose now that $C \subset \mathbb{P}^3$ is a general rational quartic curve. Describe the flat limit of the family of cycles $\Gamma_{C_t} \subset \mathbb{G}(1,3)$, and in particular show that it has not any embedded component.

Solution to Exercise ??: Since all cycles Γ_{C_t} are Cartier divisors, their flat limit will be a Cartier divisor too (whose defining polynomial will just be the limit in the projective space $\mathbb{P}^N$ of the linear series they belongs to). But now, every cycle $\Sigma_1(\overline{p_i, q})$ is a Cartier divisor as well, so is their union; hence, this has to be the flat limit, with no embedded components. This holds more in general for any curve degenerating to a union lines through the same point. □

Exercise 1.69. **??** Let $C \subset \mathbb{P}^r$ be a smooth curve. Show that the rational map $\varphi : C^{(2)} \to \mathbb{G}(1,r)$ sending a pair of distinct points $p, q \in C$ to the line $\overline{p, q}$ actually extends to a regular map on all of $C^{(2)}$ by sending the pair $2p$ to the projective tangent line $\mathbb{T}_p C$. Use this to show that the image of φ coincides with the locus of lines $L \subset \mathbb{P}^r$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2.

Solution to Exercise ??: Let L be any line in the closure of the image of $C^{(2)} \setminus \Delta$; this arises as limit of lines $L_t = \overline{p_1(t), p_2(t)}$ for $p_1(t), p_2(t)$ approaching the same point p along C for $t = 0$. Let's show that the limiting line can only be $\mathbb{T}_p L$; this, combined with the fact that $C^{(2)}$ is smooth along Δ, will prove that the map is regular on all of $C^{(2)}$. But now, the line L_t contains the two points $p_1(t), p_2(t)$ for $t \neq 0$; its limit will thus contain the *flat* limit of the two points as t approaches 0, that is, the scheme of degree two contained in C and supported in p; this scheme uniquely determines the line $\mathbb{T}_p L$. The last sentence of the statement is now obvious. □

Exercise 1.70. **??** Show by example that the conclusion of the preceding exercise is false in general if we do not assume $C \subset \mathbb{P}^r$ smooth. Is it still true if we allow C to have mild singularities, such as nodes?

Solution to Exercise ??: The first conclusion of the previous exercise, about the map extending to a regular one all over $C^{(2)}$, is false even if the singularity is al mild as possible; in fact, this is false for any nodal curve: all lines through the node n appear as limit, so the doesn't extend to a regular map at the point (n, n) of $C^{(2)}$; note that in case of a cusp the map does extend, as in case of any unibranch singularity. The second conclusion of Exercise **??**, about the closure of the image of $C^{(2)}$ is the locus of lines having degree of intersection at least 2, fails for cusps: in fact, this locus includes also all lines through the singular point in the plane in which the cusp is (locally) contained, that are not in this closure. This assertion is true though if we allow only nodal singularities (this can be proved using a locally analytic picture). □

Exercise 1.71. **??** Similarly, show by example that the conclusion of Exercise **??** is false if we consider higher-dimensional secant planes: for example, the image of the map

$$\varphi : C^{(3)} \to \mathbb{G}(2, r)$$
$$p + q + r \mapsto \overline{p, q, r}$$

need not coincide with the locus of 2-planes $\Lambda \subset \mathbb{P}^r$ whose scheme-theoretic intersection with C has degree at least 3.

Solution to Exercise ??: Note that the map extends to a regular map to all triples of points not all coincident, by Exercise **??**; to extend it further, let's follow step by step the solution of Exercise **??**: the only point that we cannot extend to this exercise, is that the scheme of the degree 3 contained in C and supported in a point need not to determine uniquely a plane. For points called inflexionary (we will see later in the book lots of examples of them) this scheme is contained in a line, so that all planes containing the line are actually intersecting the curve is degree equal to three; so, the map extends to the points $3p$, and the image is the locus of planes with degree 3 intersection, if and only if there are not inflexionary points; this happens quite rarely though, only for some very special curves such as rational normal curves. Note that if the curve lies in $\mathbb{P}^n$ with $n = 6$ or higher, in the presence of an inflexionary points these extra planes with degree 3 intersection consist of a component (a $\mathbb{P}^{n-2}$) that is higher dimensional than the image of $C^{(3)}$; so, the closure of the image can't be the whole locus. □

Exercise 1.72. **??** Show that the smooth locus of $S = Sec_2(C)$ contains the locus of lines $L \subset \mathbb{P}^3$ such that the scheme-theoretic intersection $L \cap C$ consists of two reduced points, and for such a line L identify the tangent plane $T_L S$ as a subspace of $T_L \mathbb{G}$. (When is a tangent line to C a smooth point of $Sec_2(C)$?)

Solution to Exercise ??: Let L be a line intersecting C transversely in p and q distinct points. To prove that S is smooth at L, it's enough to show that the map $\tilde{\varphi} : C^2 \to C^{(2)} \to \mathbb{G}$ is injective on tangent spaces at the point (p, q), because on (p, q) the symmetrization map $C^2 \to C^{(2)}$ is a local isomorphism. The tangent space to C^2 at

(p, q) is

$$T_pC \oplus T_qC$$

so it's enough to show that the image of these two subspaces in $T_L\mathbb{G}$ are indipendent. Taking a vector in either of the two subspaces means for L to keep passing through one of the two points, and "moving" the other by that tangent vector. In particular, it is easy to show that

$$\varphi(T_pC \oplus 0) = \{\alpha : \tilde{L} \to V/\tilde{L} \mid \alpha(\tilde{p}) \subseteq \tilde{\mathbb{T}}_pL + \tilde{L}/\tilde{L},\ \alpha(\tilde{q}) = 0\}$$

$$\varphi(0 \oplus T_qC) = \{\alpha : \tilde{L} \to V/\tilde{L} \mid \alpha(\tilde{q}) \subseteq \tilde{\mathbb{T}}_qL + \tilde{L}/\tilde{L},\ \alpha(\tilde{p}) = 0\}$$

and this two spaces are definitely independent (because the intersection is zero), so L is a smooth point of S; the tangent space will be the sum of the two above, that means,

$$T_LS = \{\alpha : \tilde{L} \to V/\tilde{L} \mid \alpha(\tilde{p}) \subseteq \tilde{\mathbb{T}}_pL + \tilde{L}/\tilde{L},\ \alpha(\tilde{q}) \subseteq \tilde{\mathbb{T}}_qL + \tilde{L}/\tilde{L}\}.$$

To answer the last question, let L be a not inflexionary tangent to C, with no other intersection, and osculating plane H; a slight modification of the previous argument shows that φ is still injective on tangent spaces, and

$$T_LS = \{\alpha : \tilde{L} \to V/\tilde{L} \mid \alpha(\tilde{L}) \subseteq \tilde{H}/\tilde{L}\}$$

so that in the end we get that $Sec_2(C)$ is smooth at all lines whose degree of intersection with C is exactly 2. □

Exercise 1.73. **??** Use the result of the preceding Exercise to show that if C and $C' \subset \mathbb{P}^3$ are two general twisted cubic curves, then the varieties $Sec_2(C)$ and $Sec_2(C') \subset \mathbb{G}(1, 3)$ of chords to C and C' intersect transversely.

Solution to Exercise ??: Let L be in the intersection $Sec_2(C) \cap Sec_2(C')$, intersecting C in p and q and intersecting C' in p' and q'; by incidence corespondence arguments such as in Exercises **??**-**??**, we can assume all four points being different, and the intersections of the line with the curves being transverse. Calling $H = \mathbb{T}_pC + L$, $K = \mathbb{T}_qC + L$, $H' = \mathbb{T}_{p'}C' + L$ and $K' = \mathbb{T}_{q'}C' + L$, by similar arguments we can assume these 4 planes being different. By the same argument as in Exercise **??**, we can also assume the two cross ratios p, q, p', q' and $H/L, K/L, H'/L, K'/L$ being different. Then, as in Exercise **??**, the claim follows. □

Exercise 1.74. **??** Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve, and let L and $M \subset \mathbb{P}^3$ be general lines.

(a) Find the number of chords to C meeting both L and M by applying the result above; and

(b) Verify this count by considering the product morphism

$$\pi_L \times \pi_M : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

(where $\pi_L, \pi_M : C \to \mathbb{P}^1$ are the projections from L and M) and comparing the arithmetic and geometric genera of the image curve.

Solution to Exercise ??: By indetermined coefficients intersecting with σ_2 and $\sigma_{1,1}$, and explicit evaluations on tangent spaces, the class of $Sec_2(C)$ in $A^2(\mathbb{G})$ is

$$[Sec_2(C)] = \left(\binom{d-1}{2} - g\right)\sigma_2 + \binom{d}{2}\sigma 1, 1.$$

Intersecting with two general cycles σ_1, and again checking on tangent spaces that the intersection is transverse, we get that the intersection consists of

$$(d-1)^2 - g$$

reduced points. About point (b), the class in $\mathbb{P}^1 \times \mathbb{P}^1$ of the image of C is (d, d), because composing with any of the two projections onto $\mathbb{P}^1$ we have a degree d cover; the arithmetic genus of the image, by adjunction formula, is then $(d-1)^2$. Comparing it with the geometric genus g, we get that the sum of all delta invariants of singularities of the curve is $(d-1)^2 - g$; supposing it has only nodal singularities, this means that there are exactly that many nodes. But now, nodes in the image of C correspond exactly to chords to C meeting L and M, so the claim follows. □

Exercise 1.75. **??** Let $C \subset \mathbb{P}^3$ be a smooth, irreducible nondegenerate curve of degree d, and let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^3$ be the family of curves specializing C to a scheme supported on the union of lines joining a point $p \in \mathbb{P}^3$ to the points of a plane section of C, as constructed in Section **??**. Show that C_0 may have an embedded point at p, and that the multiplicity of this embedded point may depend on the genus of the curve C, by considering the examples of curves of degrees 4 and 5.

Solution to Exercise ??: We will examine only the case of degree 4, and leave the degree 5 case to the reader. First, note that the specialization C_0 is going to have Hilbert polynomial equal to that of C. Let C be a rational quartic nondegenerate space curve: its Hilbert polynomial is $4m + 1$; the union of 4 lines meeting at a point, no three coplanar, is the complete intersection of 2 quadric cones, so its Hilbert polynomial can be computer in this way, to get $4m$, in the limit, then, there is an embedded point of multiplicity one at the node (the embedded point has to be on the singular locus; for a reference about this, check **?**], pag. 763). If as repelling plane we choose a plane containing a trisecant line, we get a configuration of lines in which three of them are coplanar; the Hilbert polynomial of this configuration is then $4m - 1$, so we will have an embedded point of multiplicity 2. If C is elliptic of degree 4, however (so, already the complete intersection of two quadrics!), the Hilbert polynomial of C is $4m$, so in the limit there won't be any embedded point. □

Exercise 1.76. ?? In the situation of the preceding problem, let $Sec_2(C_t) \subset \mathbb{G}(1,3)$ be the locus of chords to C_t for $t \neq 0$. Suppose that the degree of C is 4. Show that the component $\Sigma_2(p)$ will be in the flat limit with multiplicity depending on the genus of C.

Solution to Exercise ??: It is easy to show that the limit of $Sec_2(C_t)$ is supported in the union of the cycles

$$\Sigma_2(p) \cup \bigcup_{i,j} \Sigma_{1,1}(\overline{pq_iq_j})$$

now, using the formula obtained in Exercise **??** for the class of $[Sec_2(C_t)]$, we get

$$[Sec_2(C_t)] = (3-g)\sigma_2 + 6\sigma_{1,1}$$

that proves that $\Sigma_2(p)$ appears with mutiplicity 3 if C is rational, and 2 if C is elliptic. Note that this multiplicity is not the same as the one of the embedded point at p in the limit of the curves C_t! □

Exercise 1.77. ?? Again, suppose $C \subset \mathbb{P}^3$ is any curve of degree d; choose a general plane $H \subset \mathbb{P}^3$ and point $p \in \mathbb{P}^3$ and consider the one-parameter group $\{A_t\} \subset PGL_4$ with repellor point p and attractor plane H—that is, choose coordinates $[Z_0, \dots, Z_3]$ on $\mathbb{P}^3$ such that $p = [0,0,0,1]$ and H is given by $Z_3 = 0$, and consider for $t \neq 0$ the automorphisms of $\mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Let $C_t = A_t(C)$, and for $t \neq 0$ let $Sec_2(C_t) \subset \mathbb{G}(1,3)$ be the locus of chords to C_t. Show that the Schubert cycle $\Sigma_{1,1}(H)$ appears as a component of multiplicity $\binom{d}{2}$ in the limiting scheme $\lim_{t\to 0} Sec_2(C_t)$. (Hint: let $\Psi \subset \mathbb{A}^1 \times G$ be the closure of the family

$$\Psi^\circ = \{(t,L) \mid t \neq 0 \text{ and } L \in Sec_2(C_t)\},$$

and show that if $L \subset H$ is a general line, then in a neighborhood of the point $(0,L) \in \mathbb{A}^1 \times G$, the family Ψ consists of the union of $\binom{d}{2}$ smooth sheets, each intersecting the fiber $\{0\} \times G$ transversely in the Schubert cycle $\Sigma_{1,1}(H)$.)

Solution to Exercise ??: Let L be a general line in H, and let C_0 be the curve (of degree d as well) in H that is the flat limit of the curves C_t; L will meet the curve transversely in d points that we will label $p_1, \dots, p_d$. We will indicate by $p_i(\epsilon_i, t)$ points that are neighbor to p_i in the surface obtained glueing the curves C_t; the parameter

t will determine the curve C_t, the parameter ϵ_i will rapresent a movement inside a single fiber C_t, and $p_i = p_i(0,0)$; note that the surface is smooth around p_i, with ϵ_i and t as local parameters. Let now L_s be a family of lines in Ψ parametrized by a parameter s such that $L_s \in \Psi_{t(s)}$ for a certain analytic function $t(s)$ such that $t(0) = 0$, and $L_0 = L$. The intersection $L_s \cap C_{t(s)}$ has degree 2 for $t(s) \neq 0$, so for $s = 0$ the intersection has to be two reduced points p_i ad p_j. So, $C_{t(s)}$ contains points $p_i(s) = p_i(\epsilon_i(s), t(s))$ and $p_j(s) = p_j(\epsilon_j(s), t(s))$ whose limit are respectively p_i and p_j, and $L_s = \overline{p_i(s), p_j(s)}$. So, we get the following $\binom{d}{2}$ sheets

$$\Psi_{i,j} = \{\overline{p_i(\epsilon_i, t), p_j(\epsilon_j, t)} \in G \mid \epsilon_i, \epsilon_j, t \text{ small enough}\}$$

that are smooth because $\epsilon_i, \epsilon_j, t$ are local parameters around L, and intersect transversely the zero fiber because t is a local parameter. □

Exercise 1.78. **??** Let C and $C' \subset Q \subset \mathbb{P}^3$ be general twisted cubic curves lying on a smooth quadric surface Q, of types $(1,2)$ and $(2,1)$ respectively. Show that the intersection $Sec_2(C) \cap Sec_2(C')$ of the corresponding cycles of chords is transverse.

Solution to Exercise ??: Let L be a common chord; it can't lie inside Q, otherwise it would meet one of the cubics in only one point; the intersection $L \cap Q$ is then composed of two points p and q; for L to be a common chord for C and C', p and q need to be among the 5 transverse points of intersection in $C \cap C'$. Last thing to check are tangent spaces: following Exercise **??**, and by the fact that $L, \mathbb{T}_pC$ and $\mathbb{T}_pC'$ are not coplanar (otherwise, L would be tangent to Q) and similarly for q, we get that the tangent spaces are transverse. This gives another proof that the intersection of the two cycles is 10, coming from the fact that

$$[Sec_2(C)] = \sigma_2 + 3\sigma_{1,1},$$

as seen in Exercise **??**. □

Exercise 1.79. **??** Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $T(C) \subset \mathbb{G}(1,3)$ be the locus of its tangent lines. Find the class $[T(C)] \in A^3(\mathbb{G}(1,3))$ of $T(C)$ in the Grassmannian $G(1,3)$.

Solution to Exercise ??: Let L be a line that is tangent to C at a point p, with osculating plane H; using a local parameter t for C in a neighborhood of p, one can see that

$$T_L T(C) = \{\alpha : \tilde{L} \to V/\tilde{L} \mid \alpha(\tilde{p}) = 0,\ \alpha(\tilde{L}) \subset \tilde{H}/\tilde{L}\}.$$

So, when we intersect with a general cycle σ_1, the intersection will be transverse. So, we are now reduced to find how many lines are tangent to C and meet a general line M. M is disjoint from C, let's project away from M; the resulting map $C \to \mathbb{P}^1$ will be a degree d cover of $\mathbb{P}^1$, with $2d + 2g - 2$ ramification points, by Riemann-Hurwitz theorem; these points correspond to planes containing L that are tangent to C, that means, containing a line in $T(C)$. The degree of intersection is then $2d + 2g - 2$, so that we have

$$[T(C)] = (2d + 2g - 2)\sigma_{2,1}.$$

□

Exercise 1.80. **??** Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $S \subset \mathbb{P}^3$ be a general surface of degree e. How many tangent lines to C are tangent to S?

Solution to Exercise ??: Let L be a line tangent to S at a point p, such that $\mathbb{T}_p S = H$; using local parameters u, v for S around p, one can show that

$$T_L T(S) = \{\alpha : \tilde{L} \to V/\tilde{L} \mid \alpha(\tilde{p}) \subset \tilde{H}/\tilde{L}\}.$$

Let's find its class in $A^*(\mathbb{G}(1,3))$; at first, the intersection with a general $\sigma_{2,1}$ is clearly transverse; so, we need to find lines in a general plane, through a general point, and tangent to S; the plane section of S will be a degree e smooth curve, of genus $\binom{e-1}{2}$; projecting from a point, we get a degree e cover of $\mathbb{P}^1$, with

$$2e + 2\binom{e-1}{2} - 2 = e^2 - e$$

ramification points; the class is then

$$[T(S)] = (e^2 - e)\sigma_1.$$

Let L be also tangent to the curve C, at a point q, with osculating plane K. To have the intersection $T(C) \cap T(S)$ transverse at L, we need to have $p \neq q$ and $H \neq K$; let's show that from the generality of S, we can deduce both conditions; at first, we can impose that C and S meet transversely: this will avoid the situation in which $p = q$. We can also impose that the intersection between the dual surface S^* and the curve of osculating planes C^* are transverse in $\mathbb{P}^{3*}$; this will imply that $H \neq K$. So, we just need to intersect the cycles, getting

$$\#(T(C) \cap T(S)) = deg([T(C)] \cdot [T(S)]) = (2d + 2g - 2)(e^2 - e).$$

□

1.4 Chapter 4

Exercise 1.81. **??** Use the description of the points of the Schubert cells given in Theorem **??** to show that Theorem **??** holds at least set-theoretically.

Solution to Exercise ??: Let's pick a basis and a flag such that

$$V_i = < e_1, \ldots, e_i >$$

as in Theorem **??**. By the transitive action of $PGL(V)$ on flags (and on Plücker hyperplanes) everything will follow for every flag. Now, we want to say that matrices such as in the pictures of the proof of Theorem **??** are characterized by the vanshing of certain minors; this is easy to see in the pictures, let's make it precise. Let Σ_a be a Schubert cell, and Λ an element of it; it's easy to show that, for any basis we choose of Λ, in the representing matrix we will have zero minors for every choice of indices $i_1 < \ldots < i_k$ such that for at least one index i_j we have $i_j > n - k + j - a_j$. This can be shown choosing a suitable basis of Λ such an in the proof of Theorem **??** (considering the induced flag on Λ) and noticing that the condition is equivalent for the matrix to be block lower triangular with a zero on the diagonal. Now we just need to show that the vanishing locus in $G(k, n)$ of such coordinates is exactly the Schubert cell. Let Λ be a subspace satisfying these conditions, let's consider the induced flag, and let's pick a basis such that the matrix is again in the form above; the fact that all minors with $i_1 > n - k + 1 - a_1$ are zero, means that in the first row the last $k - 1 + a_1$ entries are zero, so that $dim(\Lambda \cap V_{n-k+1-a_1}) \geq 1$; working in the same way with every row, we actually get $\Lambda \in \Sigma_a$, and the claim follows (set theoretically). □

Exercise 1.82. **??** Let $X \subset G(2, 4)$ be an irreducible surface. As we observed in the preceding problem, we can write

$$[X] = \gamma_2\sigma_2 + \gamma_{1,1}\sigma_{1,1} \in A^2(G(2, 4)).$$

Show that if $\gamma_2 = 0$ then $\gamma_{1,1} = 1$. (In general, it's not known what pairs $(\gamma_2, \gamma_{1,1})$ occur!)

Solution to Exercise ??: Let's consider the variety Y in $\mathbb{P}^3$ swept out by lines in X; its dimension should be 3 and its degree given by the intersection number $[X] \cdot \sigma_2$ (or a divisor of it); in our case, this number is 0; this can only mean that Y is indeed a surface, and is irreducible (because $Y = \beta(\alpha^{-1}X)$ and α has irreducible fibers). This surface is ruled by lines, and for any point of Y there is a positive dimensional family of lines through it and contained in Y (otherwise Y would be 3 dimensional); let's prove that the only possibility for that is a plane (that would prove $\gamma_{1,1} = 1$); let p a point of Y; by hypothesis, a cone with p as vertex and over a curve C is contained in Y; by irreducibility of Y, C must be irreducible as well, and Y has to be exactly this cone; but now if we take another point q on Y, there is only one line contained in Y through it, unless C is a line, and Y is a plane. □

Exercise 1.83. **??** Let $S \subset \mathbb{P}^4$ be a surface of degree d, and $\Gamma_S \subset \mathbb{G}(1, 4)$ the variety of lines meeting S.

(a) Find the class $\gamma_S = [\Gamma_S] \in A^1(\mathbb{G}(1, 4))$.

(b) Use this to answer the question: if $S_1, \ldots, S_6 \subset \mathbb{P}^4$ are surfaces of degrees $d_1, \ldots, d_6$, how many lines in $\mathbb{P}^4$ will meet all six?

Solution to Exercise ??: Assuming characteristic zero, by Kleiman's theorem the class is $\gamma\sigma_1$, where γ is the number of points in the intersection with a general $\sigma_{3,2}$ class; this means, considering lines in $\mathbb{P}^4$ that meet S, and that are in a pencil of lines through a general point and contained in a general plane P. The points of intersection of S and P are exactly d, and so we get d such lines; the class is $d\sigma_1$. Without getting involved in transversality arguments and without using general translates of the surfaces (see Exercises **??**,**??**,**??**,**??** for a similar treatment), this tells us that the *expected* number of lines meeting six such surfaces are $d_1d_2d_3d_4d_5d_6 \times deg(\sigma_1^6) = 5d_1d_2d_3d_4d_5d_6$. □

Exercise 1.84. **??** Let $C \subset \mathbb{P}^4$ be a curve of degree d, and $\Gamma_C \subset \mathbb{G}(1,4)$ the variety of lines meeting C.

(a) Find the class $\gamma_C = [\Gamma_C] \in A^2(\mathbb{G}(1,4))$.
(b) Use this to answer the question: if C_1, C_2 and $C_3 \subset \mathbb{P}^4$ are curves of degrees d_1, d_2 and d_3, how many lines in $\mathbb{P}^4$ will meet all three?

Solution to Exercise ??: Let's assume again characteristic zero, and talk about *expected* number of lines; again, we suggest the reader interested in all the details to follow step by step Exercises **??**,**??**,**??**,**??**. Now, the class of Γ_C is of the kind

$$\gamma_2\sigma_2 + \gamma_{1,1}\sigma_{1,1},$$

and we can find the coefficients intersecting with general $\sigma_{3,1}$ and $\sigma_{2,2}$ classes. Intersecting with $\sigma_{3,1}$ means finding the number of lines meeting the curve, contained in a $\mathbb{P}^3$, and through a fixed point; the intersection of C with the $\mathbb{P}^3$ is d points, so we have d lines. About the intersection with $\sigma_{2,2}$, we lok for lines meeting C and contained in a general 2-plane, and we don't have any. The class is that $d\sigma_2$. If we intersect three such classes, we get

$$d_1d_2d_3 \cdot deg(\sigma_2^3) = d_1d_2d_3 \cdot deg((\sigma_{3,1} + \sigma_{2,2}) \cdot (\sigma_2)) = d_1d_2d_3.$$

□

The following exercise **??** is the first of a series regarding the variety $T_1(S)$ of lines tangent to a surface S in $\mathbb{P}^n$. More will follow in Exercises **??**, **??** and **??**.

Exercise 1.85. **??** Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d whose general hyperplane section is a curve of genus g and $T_1(S) \subset \mathbb{G}(1,n)$ the variety of lines tangent to S. To find the class of the cycle $T_1(S)$, we need the intersection numbers $[T_1(S)] \cdot \sigma_3$ and $[T_1(S)] \cdot \sigma_{2,1}$; find the latter.

Solution to Exercise ??: For every point $p \in S$, we have a 2 dimensional tangent space, so a $\mathbb{P}^1$ of lines through that point tangent to S at p. This means that $T_1(S)$ is a 3 dimensional variety in $\mathbb{G}(1,n)$, so that the cycle is going to be of the form

$$\alpha\sigma_{n-1,n-4} + \beta\sigma_{n-2,n-3}.$$

To find the coefficients we have to intersect with general σ_3 and $\sigma_{2,1}$ cycles respectively; a general $\sigma_{2,1}$ cycle will be composed by lines contained in a general $\mathbb{P}^{n-1}$, touching a general $\mathbb{P}^{n-3}$ inside it. The lines of $T_1(S)$ contained in a $\mathbb{P}^{n-1}$ are lines tangent to a general hyperplane section $H \cap S$ of S, that is, a curve of degree d and genus g in $\mathbb{P}^{n-1}$. The number of such tangent lines meeting a $\mathbb{P}^{n-3}$ is the same as the number of ramification points of the projection of the curve $H \cap S$ away from $\mathbb{P}^{n-3}$ onto $\mathbb{P}^1$; by Riemann-Hurwitz theorem, this number is $2d + 2g - 2$, so that

$$[T_1(S)] \cdot \sigma_{2,1} = 2d + 2g - 2.$$

□

Exercise 1.86. ?? Let $Z \subset \mathbb{G}(k,n)$ be a variety of dimension m, and consider the variety $X \subset \mathbb{P}^n$ swept out by the linear spaces corresponding to points of Z: that is,

$$X = \bigcup_{[\Lambda]\in Z} \Lambda \subset \mathbb{P}^n.$$

For simplicity, assume that a general point $x \in X$ lies on a unique k-plane $\Lambda \in Z$.

(a) show that X has dimension $k+m$ and degree the intersection number $\deg(\sigma_m \cdot [Z])$.
(b) Show that this is not in general the degree of Z.

Solution to Exercise ??: Let $\Phi_X \subset \mathbb{G}(k,n) \times \mathbb{P}^n$ be the incidence correspondence

$$\Phi_X = \{([\Lambda], x) \mid [\Lambda] \in Z, x \in \Lambda\}$$

just obtained as the preimage of Z from the projection onto $\mathbb{G}(k,n)$ (so it has dimension $k+m$); the variety X is obtained as the image by Φ_X the projection onto $\mathbb{P}^n$; now this projection on Φ_X is generically one to one from the hypothesis on the general point x of X, so X is a closed subvariety of $\mathbb{P}^n$ of dimension $k+m$. Its degree will be given as the intersection with a general $\mathbb{P}^{n-m-k}$; note that a general such plane will intersect X in finitely many points, all lying in different Λ of Z (if two linear spaces meet in two points, they meet along a line!); so, it is the same thing as asking how many of the planes Λ of Z intersect a general $\mathbb{P}^{n-m-k}$, that is the same as finding the intersection between Z and a σ_m cycle in $\mathbb{G}(k,n)$. Note that the degree of Z as subvariety of $\mathbb{G}(k,n)$ (and then by Plücker embedding of $\mathbb{P}^N$ for some big N) is obtained intersecting with m generic hyperplanes, that means, with the cycle σ_1^m, in general very different from σ_m! □

Exercises **??**-**??** deal with the geometry of the surface described in Keynote Question (**??**), whose degree we worked out in Section **??**: the surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a general twisted cubic $C \subset \mathbb{G}(1,3)$.

Exercise 1.87. ?? To start, use the fact that the dual of $\mathbb{G}(1,3) \subset \mathbb{P}^5$ has degree 2 to show that a general twisted cubic $C \subset \mathbb{G}(1,3)$ lies on two Schubert cycles $\Sigma_1(L)$ and $\Sigma_1(M)$ for some pair of skew lines $L, M \subset \mathbb{P}^3$.

Solution to Exercise ??: A twisted cubic linearly spans a $\mathbb{P}^3$, so the hyperplanes in $\mathbb{P}^5$ containing it are going to be a pencil in the dual space $\mathbb{P}^{5*}$ (that can be seen as the Grasmannian $\mathbb{G}(4,5)$). Remember that hyperplane sections of $\mathbb{G}(1,3)$ correspond to a σ_1 cycle if and only if they give a singular section, that means, if they come from hyperplanes tangent to $\mathbb{G}(1,3)$; in fact, a singular hyperplane section is a cone over a quadric surface, so that the vertex determines the line $[L]$ such that the hyperplane section is $\Sigma_1(L)$. So, σ_1 cycles come from hyperplanes in $\mathbb{P}^{5*}$ that are tangent to $\mathbb{G}(1,3)$, that means, the dual variety of $\mathbb{G}(1,3)$; this has degree 2, so the pencil of hyperplanes containing the general twisted cubic will intersect it twice; in sum, the twisted cubic is contained in two Schubert cycles $\Sigma_1(L)$ and $\Sigma_1(M)$ (with L and M skew because of the generality of C. □

Exercise 1.88. ?? Show that for skew lines L and $M \subset \mathbb{P}^3$, the intersection $\Sigma_1(L) \cap \Sigma_1(M)$ is isomorphic to $L \times M$ via the map sending a point $[\Lambda] \in \Sigma_1(L) \cap \Sigma_1(M)$ to the pair $(\Lambda \cap L, \Lambda \cap M) \in L \times M$, and that it is the intersection of $\mathbb{G}(1,3)$ with the intersection of the hyperplanes spanned by $\Sigma_1(L)$ and $\Sigma_1(M)$.

Solution to Exercise ??: As seen in the previous exercise, $\Sigma_1(L)$ is a cone over a smooth quadric surface; now, every hyperplane section of it that does not contain the vertex $[L]$ will give rise to a smooth quadric surface; but now, intersecting with $\Sigma_1(M)$ means considering an hyperplane section, and the fact that L and M are skew means that $L \notin \Sigma_1(M)$, so that $\Sigma_1(L) \cap \Sigma_1(M)$ is indeed isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; it is very to show that the map to $L \times M$ realizes this isomorphism, because it is bijective and has a bijective inverse. □

Exercise 1.89. ?? Finally, suppose that $C \subset \Sigma_1(L) \cap \Sigma_1(M)$ is a twisted cubic curve. Using the fact that its bidegree in $\Sigma_1(L) \cap \Sigma_1(M) \cong L \times M \cong \mathbb{P}^1 \times \mathbb{P}^1$ (possibly after switching factors) is (2, 1), show that for some degree 2 map $\varphi : L \to M$, the family of lines corresponding to C may be realized as the locus

$$C = \{\overline{p, \varphi(p)} \mid p \in L\}.$$

Show correspondingly that the surface

$$X = \bigcup_{[L] \in C} L \subset \mathbb{P}^3$$

swept out by the lines of C is a cubic surface double along a line, and that it's the projection of a rational normal surface scroll $X_{1,2} \subset \mathbb{P}^4$.

Solution to Exercise ??: Given the fact that C has bidegree (2,1) on $L \times M$, projecting C onto L is going to give an isomorphism. So, C inside $L \times M$ is going to be the graph of a regular degree morphism $\varphi : L \to M$, of degree 2. So, the first part of the statement is true, we have

$$C = \{\overline{p, \varphi(p)} \mid p \in L\}.$$

Consider now, in $\mathbb{P}^4$, a rational normal surface scroll $X_{1,2}$; remember that this arises from the choice of a line E and a conic D in general position, an isomorphism $\psi : E \to D$, and considering the union of the lines joining two identified points. Let us now pick isomorphisms $\varphi_E : E \to L$ and $\varphi_D : D \to L$, in such a way the composition $\varphi_D^{-1} \circ \psi \circ \varphi_E$ is the identity on L. Let's consider then the rational map from $\mathbb{P}^4$ to $\mathbb{P}^3$ such sending E on L by φ_E, and sending D on M by $\varphi \circ \varphi_D$; this last condition is equivalent as sending the plane containing D into M, in such a way the induced morphism $L \to M$ is indeed the degree 2 cover φ. It is easy to prove (for instance, using coordinates) that this induces a unique linear rational map from $\mathbb{P}^4$ to $\mathbb{P}^3$, defined everywhere except for a point p in the plane of D (this rational map can be seen as well as a projection from p), and that the image of X_{12} is exactly the surface X swept out by C. This proves again that this is a degree 3 surface, and that is double along M because every point of M is image of two points of X_{12}; then, p is not in any other tangent space to X_{12}, so that the only singularity of X is M. □

$$(x_0x_3 - x_1x_2)^2 - 4(x_0x_2 - x_1^2)(x_1x_3 - x_2^2).$$

In Section **??** we calculated the number of lines meeting four general n-planes in $\mathbb{P}^{2n+1}$. In the following two exercises, we'll see another way to do this (analogous to the alternative count of lines meeting four lines in $\mathbb{P}^3$ given in Exercise **??**), and a nice geometric sidelight.

Exercise 1.90. ?? Let $\Lambda_1, \dots, \Lambda_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be four general n-planes. Calculate the number of lines meeting all four by showing that the union of the lines meeting Λ_1, Λ_2 and Λ_3 is a Segre variety $S_{1,n} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ and using the calculation of Section **??** for the degree of $S_{1,n}$.

Solution to Exercise ??: Let X be the union of all these lines; let us first show that X is the disjoint union of all the lines in $S = \cap_{i=1}^3 \Sigma_n(\Lambda_i)$, that are parametrized by $\Lambda_1 \cong \mathbb{P}^n$; in fact, consider a point $p \in \Lambda_1$: projecting away from it, the images of Λ_2 and Λ_3 will intersect in one single point: this means that there is only one line L in S passing through p; it is easy to show then that this lines are all disjoint. This provides also a map $X \to \mathbb{P}^n$, whose fibers are isomorphic to $\mathbb{P}^1$. This map has three independent sections, corresponding to the embeddings $\Lambda_i \subset \mathbb{P}^n$: this can only mean that X is the trivial $\mathbb{P}^1$ fibration on $\mathbb{P}^n$, that means just the product $\mathbb{P}^1 \times \mathbb{P}^n$. To prove that this is indeed embedded in $\mathbb{P}^{2n+1}$ as the Segre embeddings, we need to prove that all fibers $\mathbb{P}^n$ are embedded as projective subspaces of $\mathbb{P}^{2n+1}$. This might be a little complicated to prove directly, so we will do it using coordinates: using the fact that the projective linear group PGL_{2n+2} acts transitively on general triples on n-planes $(\Lambda_1, \Lambda_2, \Lambda_3)$, we just need to prove it for one specific situation. Using the three planes

$$\Lambda_1 = (x_0, x_1, \ldots, x_n)$$

$$\Lambda_2 = (x_{n+1}, x_{n+2}, \ldots, x_{2n+1})$$

$$\Lambda_3 = (x_0 - x_{n+1}, \ldots, x_n - x_{2n+1})$$

it is easy to see that the Segre variety is the locus where the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_n \\ x_{n+1} & x_{n+2} & \ldots & x_{2n+1} \end{pmatrix}$$

has rank 1 (meaning the vanishing locus of all 2×2 determinants), that is a Segre variety $S_{1,n}$. □

Exercise 1.91. ?? By the preceding exercise, we can associate to a general configuration $\Lambda_1, \ldots, \Lambda_4$ of k-planes in $\mathbb{P}^{2k+1}$ an unordered set of $k+1$ cross-ratios. Show that two such configurations $\{\Lambda_i\}$ and $\{\Lambda_i'\}$ are projectively equivalent if and only if the corresponding sets of cross-ratios coincide.

Solution to Exercise ??: Remember that the action of PGL_{2n+2} on general triples $(\Lambda_1, \Lambda_2, \Lambda_3)$ of n-planes transitive; furthermore, the stabilizer of a general triple is isomorphic to PGL_{n+1}; in fact, this stabilizer would keep fixed the Segre variety $S_{1,n}$ of the previous exercise, whose stabilizer is $PGL_2 \times PGL_{n+1}$; but, in this $\mathbb{P}^1 \times \mathbb{P}^n$ the three $\mathbb{P}^n$ fibers corresponding to $\Lambda_1, \Lambda_2, \Lambda_3$ have to remain fixed, so that in the product $PGL_2 \times PGL_{n+1}$ the first coordinate has to be the identity. Note that this proves that the set of linear automorphism sending a general triple $(\Lambda_1, \Lambda_2, \Lambda_3)$ into another general triple $(\Lambda_1', \Lambda_2', \Lambda_3')$ is isomorphic to PGL_{n+1} as well. Now, for an element of PGL_{n+1} to keep fixed also a fourth general plane Λ_4, it is necessary and sufficient to fix the $n+1$ points of intersection of Λ_4 with $S_{1,n}$; we can represent these points in $\mathbb{P}^1 \times \mathbb{P}^n$ as $\{(\lambda_i, p_i)\}_{i=1}^{n+1}$ and consider which $n+1$-tuples are conjugate by the action of PGL_{n+1}; but now PGL_{n+1} acts trivially on the first coordinates λ_i and it acts transitively on (general) $n+1$-tuples of points in $\mathbb{P}^n$. Collecting everything, we proved that two general 4-tuples of n-planes are projectively equivalent if and only if the set $\{p_i\}_{i=1}^{n+1}$ of $n+1$ points in $\mathbb{P}^1$ is the same; these points can be seen as cross ratios of the points of intersection with the 4 planes of the $n+1$ lines meeting them. □

The next two exercises deal with the example of dynamic specialization given in Section **??**, and specifically with the family Φ of cycles described there.

Exercise 1.92. **??** Show that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$.

Solution to Exercise ??: In Section **??**, we proved that the support of Φ_0 is *contained* in $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$. Let now N be a general line in $\Sigma_{2,2}(P)$, that means, a line contained in P; this line meets L in a point q, and M_0 in a point q_0; the point q_0 is a limit of points $q_t \in M_t$, so considering the lines $N_t = \overline{qq_t}$ we obtain a family of lines whose limit is N. Let now N' be a general line in $\Sigma_{3,1}(p_0, H_0)$, that means a line contained in the hyperplane H_0 and through the point p_0; consider now the plane $K = \overline{LN'}$, and a point q' in N' different from p_0; consider now any family of points q'_t having as limit q' such that $q'_t \in H_t$, and consider the family of planes $K_t = \overline{Lq'_t}$. Now, every line M_t meets the plane K_t in a point r_t, because both M_t and K_t lie in the same three space H_t; note that we have $r_0 = p_0$. Now, we just need to consider the family of lines $N'_t = \overline{q'_t r_t}$; they have N' as limit, and N'_t meets M_t in r_t, and meets L because it lies in the plane K_t; N' is then in the limit. □

Exercise 1.93. **??** Verify the last assertion made in the calculation of σ_2^2; that is, show that Φ_0 has multiplicity 1 along each component. [Hint: argue that by applying a family of automorphisms of $\mathbb{P}^4$ we can assume that the plane H_t is constant and use the calculation of the preceding chapter.]

Solution to Exercise ??: As in the calculations for the previous chapter, this comes from the fact that general lines in $\Sigma_{2,2}(P)$ and $\Sigma_{3,1}(p_0, H_0)$ are limit of a single line in neighbor fibers Φ_t. □

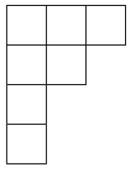
Exercise 1.94. **??** A further wrinkle in the technique of dynamic specialization is that to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages. To see an example of this, use dynamic specialization to calculate the intersection σ_2^2 in the Grassmannian $\mathbb{G}(1, 5)$. [Hint: you have to let the two 2-planes specialize first the a pair intersecting in a point, then to a pair intersecting in a line.]

Solution to Exercise ??: Let's consider two planes P, Q; if we have Q directly degenerating to a plane Q_0 meeting P along a line L, we would have $\Sigma_2(P) \cap \Sigma_2(Q_0)$ containing a component with dimension too high, that is $\Sigma_3(L)$. As suggested, let us now perform the specialization in two steps; the first has the plane Q degenerating to a plane Q' meeting P in a point p; then, the intersection of the two cycles splits in two irreducible components, one of which is $\Sigma_4(p)$ and the other one Ψ whose general point is a line meeting P and Q' at two different points (thus contained in the 4-plane $\overline{PQ'}$); in the same fashion as in the previous exercise, it is possible to prove that both components appear with multeplicity one. The second step of the specialization will be moving Q' in a plane Q'' meeting P along a line L. The component Ψ will then break in the components $\Sigma_{3,1}(L, \overline{PQ'})$ and $\Sigma_{2,2}(\overline{PQ''})$, that again can be show they appear with multiplicity 1 each; we then found

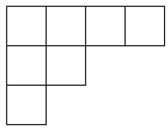
$$\sigma_2^2 = \sigma_4 + \sigma_{3,1} + \sigma_{2,2}.$$

□

Exercise 1.95. ?? Suppose that the Schubert class $\sigma_a = \in A(G(k,n))$ corresponds to the Young diagram Y in a $k \times (n-k)$ box B. Show that under the duality $G(k,n) \cong G(n-k,n)$, the class σ_a is taken to the Schubert class σ_b corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-southeast. For example if

$\sigma_{3,2,1,1} \in A(G(4,7)) \quad \longleftrightarrow$ 

then the corresponding Schubert cycle in $G(3,7)$ is

$\sigma_{4,2,1} \in A(G(3,7)) \quad \longleftrightarrow$ 

Solution to Exercise ??: The solution of this exercise is left to the reader. □

Exercise 1.96. ?? Let $i : G(k,n) \to G(k+1, n+1)$ and $j : G(k,n) \to G(k, n+1)$ be the inclusions obtained by sending $\Lambda \subset K^n$ to Λ and to the span of Λ and e_{n+1} respectively. Show that the map $i^* : A^d(G(k+1,n+1)) \to A^d(G(k,n))$ is a monomorphism if and only if $n-k \geq d$, and that $j^* : A^d(G(k,n+1)) \to A^d(G(k,n))$ is a monomorphism if and only if $k \geq d$. (Thus, for example, the formula

$$\sigma_1^2 = \sigma_2 + \sigma_{11},$$

which we established in $A(\mathbb{G}(1,3))$, *holds true in every Grassmannian.*)

Solution to Exercise ??: Consider a Schubert variety $\Sigma_a(\mathcal{V})$ for a general flag $\mathcal{V}$ in $G(k, n+1)$; its preimage in $G(k,n))$ by j correspond to

$$\Sigma_a(\mathcal{V}) \cap \Sigma_{1,\ldots,1}(K^n).$$

As Schubert cycle in $G(k,n))$, this correspond to

$$\Sigma_a(\mathcal{V} \cap K^n)$$

where $\mathcal{V} \cap K^n$ is the flag obtained intersecting all elements of $\mathcal{V}$ with the hyperplane K^n. This gives rise to a cycle σ_a, unless one of the elements of $\mathcal{V}$ is a point so that the intersection with K^n is empty, and thus the whole image. The condition can be stated as the Young diagram of the partition a to have no rows with $n-k+1$ elements, that is the same as for the partition a to make sense as Schubert class for $G(k,n)$. We thus have $j^*\sigma_a = \sigma_a$ if a has no columns with $n-k+1$ blocks, and $i^*\sigma_a = 0$ otherwise. So $j^* : A^d(G,(k,n+1)) \to A^d(G(k,n))$ is injective if and only if no partition of weight a has a row with $n-k+1$ elements, that means if $d \le n-k$.

The same argument applies for i^*; here the preimage of a Schubert variety $\Sigma_a(\mathcal{V})$ in $G(k+1,n+1)$ is

$$\Sigma_a(\mathcal{V}) \cap \Sigma_{n-k}(e_{n+1})$$

that can be seen in $G(k,n)$ as $\Sigma_a(\overline{\mathcal{V}e_{n+1}} \cap K^n)$, where $\overline{\mathcal{V}e_{n+1}} \cap K^n$ is the flag obtained considering the spans of the elements and e_{n+1}, and intersecting with K^n; again it is easy to see that if one of the elements of $\mathcal{V}$ is an hyperplane, i. e. if the partition has a column with $k+1$ elements; so, the condition on d is if there is no Young diagram with d blocks having a column with $k+1$ elements, meaning $d \le k$. □

Exercise 1.97. **??** Let $C \subset \mathbb{P}^r$ be a smooth, irreducible, nondegenerate curve of degree d and genus g, and let $S_1(C) \subset \mathbb{G}(1,r)$ be the variety of chords to C, as defined in Section **??** above. Find the class $[S_1(C)] \in A_2(\mathbb{G}(1,r))$.

Solution to Exercise ??: The variety $S_1(C)$ is 2 dimensional, so the cycle will be of the kind

$$\alpha\sigma_{r-1,r-3} + \beta\sigma_{r-2,r-2},$$

and α, β are obtained intersecting with cycles (respectively) $\sigma_{2,0}$ and $\sigma_{1,1}$. For the former, we ask how many chords to C meet a general $\mathbb{P}^{n-3}$; projecting away from that $\mathbb{P}^{n-3}$, C is mapped in a plane curve of degree d, geometric genus g, and $\binom{d-1}{2} - g$ nodes, that correspond to chords to C in $\mathbb{P}^n$. For the latter, we ask how many chords are contained in a $\mathbb{P}^{n-1}$; inside this there are d points of C, and so $\binom{d}{2}$ chords. We get then (in characteristic 0)

$$[S_1(C)] = \left(\binom{d-1}{2} - g\right)\sigma_{r-1,r-3} + \binom{d}{2}\sigma_{r-2,r-2}.$$

□

Exercise 1.98. **??** Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface, and let $T_k(Q) \subset \mathbb{G}(k,n)$ be the locus of planes $\Lambda \subset \mathbb{P}^n$ such that $\Lambda \cap Q$ is singular. Show that

$$[T_k(Q)] = 2\sigma_1.$$

Solution to Exercise ??: To find this class we have to intersect with the cycle

$$[T_k(Q)] \cdot \sigma_{n-k,n-k,\dots,n-k,n-k-1}$$

corresponding to lines in $\mathbb{G}(k,n)$ parametrizing a pencil of k-planes contained in a

$k+1$-plane Λ_{n+1} and containing an $n-1$-plane Λ_{n-1}. The intersection $Q \cap \Lambda_{n+1}$ is a smooth quadric hypersurface Q'; the number of hyperplanes in a pencil with a singular section, that is the same as being tangent to Q', is the degree of the dual of Q', that is 2; this proves that

$$[T_k(Q)] \cdot \sigma_{n-k,\dots,n-k-1} = 2$$

that means $[T_k(Q)] = 2\sigma_1$. □

Exercise 1.99. ?? Find the expression of $\sigma_{2,1}^2$ as a linear combination of Schubert classes in $A(G(3,6))$. This is the smallest example of a product of two Schubert classes where another Schubert class appears with multiplicity > 1.

Solution to Exercise ??: Note that by Pieri's formulas, no coefficient bigger than 1 can appear in the product of two Schubert cycles if one of them is of the kind $\sigma_{(1)^a}$ or σ_b; the simplest case in which this does not happen is $\sigma_{2,1}^2$. Using Giambelli's formula, $\sigma_{2,1} = \sigma_2\sigma_1 - \sigma_3$; we also have, by Pieri's formula,

$$\sigma_{2,1}\sigma_2\sigma_1 = \sigma_{3,3} + 3\sigma_{3,2,1} + \sigma_{2,2,2} \text{ and } \sigma_{2,1}\sigma_3 = \sigma_{3,2,1}$$

so that

$$\sigma_{2,1}^2 = \sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}$$

so we get a coefficient bigger than one. □

Exercise 1.100. ?? Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(2,5)$.

Solution to Exercise ??: The proof of this exercise is left to the reader. □

Exercise 1.101. ?? Let Q, Q' and Q'' be three general quadrics in $\mathbb{P}^8$. How many 2-planes lie on all three? (Try first to do this without the tools introduced in Section **??**.)

Solution to Exercise ??: By the calculations in section **??**, we need to find the intersection $(8\sigma_{3,2,1})^3$ (and invoke Kleiman's theorem); let us now focus our attention to find the triple product $\sigma_{3,2,1}^3$. Without invoking Pieri's and Giambelli's formula, we can do a first step that simplifies the problem very much. One of the conditions for a plane Λ to be in a $\sigma_{3,2,1}$ cycle is to be contained in an hyperplane H; all planes in the intersection of three general $\sigma_{3,2,1}$ cycles will then be contained in the intersection of three hyperplanes, that means, a $\mathbb{P}^5$; we can then translate the question to one about planes in $\mathbb{P}^5$; in $\mathbb{G}(2,5)$, the condition $\sigma_{3,2,1}$ becomes a $\sigma_{2,1}$, so all boils down to the intersection number $\sigma_{2,1}^3$ in $\mathbb{G}(2,5)$. Now we can use the previous exercises to get

$$\sigma_{2,1}^2 \cdot \sigma_{2,1} = (\sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}) \cdot \sigma_{2,1} = 2\sigma_{3,3,3}$$

so the degree is 2. Going back to our original problem, we have $deg(8\sigma_{3,2,1})^3 = 8^3 \cdot 2 = 1024$. □

Exercise 1.102. **??** Use Pieri to identify the degree of $\sigma_1^{k(n-k)}$ with the number of standard tableaux: that is, ways of filling in $k \times (n-k)$ matrix with the integers $1, \dots, k(n-k)$ in such a way that every row and column is strictly increasing. Then use the "hook formula" (see for example, **?**]) to show that this number is

$$\big(k(n-k)\big)! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!},$$

Solution to Exercise ??: Using Pieri's formula $k(n-k)$ times, the degree is equal to the number of ways the $k \times (n-k)$ rectangle can be "assembled" using single boxes; remembering the order in which they are assembled, they give rise to Young tableaux; remember that given a cell, its *hook* is the set of the cells directly on its right or on its left. The *hook formula* says that the number of Young tableaux of a partition λ is $|\lambda|!$ divided by the product of the sizes of all hooks. For the element in the ith row and in the jth column, its hook is composed by $n+1-i-j$ elements; taking the product in the ith row, we get $(n-k+i-1)!/(i-1)!$; collecting it all together, and shifting i by one, we get the formula above. □

Exercise 1.103. **??** Deduce Giambelli's formula in the 3×3 case—that is, the relation

$$\begin{vmatrix} \sigma_a & \sigma_{a+1} & \sigma_{a+2} \\ \sigma_{b-1} & \sigma_b & \sigma_{b+1} \\ \sigma_{c-2} & \sigma_{c-1} & \sigma_c \end{vmatrix} = \sigma_{a,b,c}$$

for any $a \geq b \geq c$—by assuming Giambelli in the 2×2 case, expanding the determinant by cofactors along the last column and applying Pieri.

Solution to Exercise ??: Expanding along the last column and applying Giambelli, we get

$$\sigma_{a+2}\sigma_{b-1,c-1} - \sigma_{b+1}\sigma_{a,c-1} + \sigma_c\sigma_{a,b}.$$

Using Pieri, we get

$$\sigma_{a+2}\sigma_{b-1,c-1} = \sum_{I_1} \sigma_{i,j,k}$$

$$I_1 = \{(i,j,k) \mid i+j+k = a+b+c,\ 0 \leq k \leq c-1 \leq j \leq b-1 \leq i \leq a+b+1\}$$

$$\sigma_{b+1}\sigma_{a,c-1} = \sum_{I_2} \sigma_{i,j,k}$$

$$I_2 = \{(i,j,k) \mid i+j+k = a+b+c,\ 0 \leq k \leq c-1 \leq j \leq a \leq i \leq a+b+1\}$$

$$\sigma_c\sigma_{a,b} = \sum_{I_3} \sigma_{i,j,k}$$

$$I_3 = \{(i,j,k) \mid i+j+k = a+b+c,\ 0 \le k \le b \le j \le a \le i \le a+b\}$$

and now it is just a combinatorial exercise to show that

$$I_1 \cup I_3 = I_2 \cup \{(a,b,c)\}.$$

□

1.5 Chapter 5

Many of the following exercises give applications of the Whitney formula and splitting principle. We will be assuming the basic facts that if

$$\mathcal{E} = \bigoplus_{i=1}^{e} \mathcal{L}_i \quad \text{and} \quad \mathcal{F} = \bigoplus_{i=1}^{f} \mathcal{M}_i$$

are direct sums of line bundles, then

$$\operatorname{Sym}^k \mathcal{E} = \bigoplus_{1 \le 1_1 \le \cdots \le i_k \le r} \mathcal{L}_{i_1} \otimes \ldots \cdots \otimes \mathcal{L}_{i_k};$$

$$\wedge^k \mathcal{E} = \bigoplus_{1 \le 1_1 < \cdots < i_k \le r} \mathcal{L}_{i_1} \otimes \ldots \cdots \otimes \mathcal{L}_{i_k}; \text{ and}$$

$$\mathcal{E} \otimes \mathcal{F} = \bigoplus_{i,j=1,1}^{e,f} \mathcal{L}_i \otimes \mathcal{M}_j.$$

Exercise 1.104. ?? Let $\mathcal{E}$ be a vector bundle of rank 3. Express the Chern classes of $\wedge^2\mathcal{E}$ in terms of those of $\mathcal{E}$ by invoking the splitting principle and the Whitney formula.

Solution to Exercise ??: Suppose $\mathcal{E}$ splits as sum of three line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ with first Chern classes $\alpha_1, \alpha_2, \alpha_3$; from the Whitney formula, we have. Then we have

$$\wedge^2\mathcal{E} = (\mathcal{L}_1 \otimes \mathcal{L}_2) \oplus (\mathcal{L}_1 \otimes \mathcal{L}_3) \oplus (\mathcal{L}_2 \otimes \mathcal{L}_3)$$

so that from the Whitney's formula we have

$$\begin{aligned} c(\wedge^2\mathcal{E}) &= (1+\alpha_1+\alpha_2)(1+\alpha_1+\alpha_3)(1+\alpha_2+\alpha_3) = \\ &= 1 + 2(\alpha_1+\alpha_2+\alpha_3) + (\alpha_1^2+\alpha_2^2+\alpha_3^2+3\alpha_1\alpha_2+3\alpha_1\alpha_3+3\alpha_2\alpha_3) + \\ &+ (\alpha_1^2\alpha_2+\alpha_1^2\alpha_3+\alpha_1\alpha_2^2+\alpha_2^2\alpha_3+\alpha_1\alpha_3^2+\alpha_2\alpha_3^2+2\alpha_1\alpha_2\alpha_3) = \\ &= 1 + 2(\alpha_1+\alpha_2+\alpha_3) + ((\alpha_1+\alpha_2+\alpha_3)^2+\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) + \\ &+ ((\alpha_1+\alpha_2+\alpha_3)(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)-\alpha_1\alpha_2\alpha_3) \end{aligned}$$

that can be rewritten using Chern classes of $\mathcal{E}$ as

$$c(\wedge^2\mathcal{E}) = 1 + 2c_1(\mathcal{E}) + c_1(\mathcal{E})^2 + c_2(\mathcal{E}) + c_1(\mathcal{E})c_2(\mathcal{E}) - c_3(\mathcal{E}).$$

By the splitting principle, this formula holds in general for every rank 3 vector bundle, and collecting by degree we get all Chern classes of $c(\wedge^2\mathcal{E})$. □

Exercise 1.105. ?? Verify your answer to the preceding exercise by observing that wedge product map

$$\mathcal{E} \otimes \wedge^2\mathcal{E} \to \wedge^3\mathcal{E} = \det(\mathcal{E})$$

yields an identification $\wedge^2\mathcal{E} = \mathcal{E}^* \otimes \det(\mathcal{E})$, and applying the formula for tensor product with a line bundle.

Solution to Exercise ??: Let us denote by c_1, c_2, c_3 the Chern classes of $\mathcal{E}$. Using the formula for the tensor product with a line bundle, we have

$$\begin{aligned} c(\mathcal{E}^* \otimes \det(\mathcal{E})) &= \sum_l c_l(\mathcal{E}^*)(1 + c_1(\det(\mathcal{E})))^{3-l} = \\ &= (1 + c_1)^3 - c_1(1 + c_1)^2 + c_2(1 + c_1) - c_3 \\ &= 1 + 2c_1 + c_1^2 + c_2 + c_1c_2 - c_3, \end{aligned}$$

that is the same as the one of $\wedge^2\mathcal{E}$; to prove that they are identified by the wedge product map, we need to prove that $\wedge^2\mathcal{E} \to \mathcal{E}^* \otimes \det(\mathcal{E})$ is an isomorphism; but by very simple linear algebra, this in an isomorphism on every fiber, so that it is globally. □

Exercise 1.106. ?? Let $\mathcal{E}$ be a vector bundle of rank 4. Express the Chern classes of $\wedge^2\mathcal{E}$ in terms of those of $\mathcal{E}$.

Solution to Exercise ??: Following the solution to Exercise **??**, we split $\mathcal{E}$ in 4 line bundles with first Chern class α_i; using the following identities of symmetric functions

$$\sum \alpha_i^3 + 7\sum \alpha_i^2\alpha_j + 8\sum \alpha_i\alpha_j\alpha_k = (\sum \alpha_i)^3 + 4(\sum \alpha_i)(\sum \alpha_i\alpha_j)$$

$$\begin{aligned} 2\sum \alpha_i^3\alpha_j + 5\sum \alpha_i^2\alpha_j^2 + 12\sum \alpha_i^2\alpha_j\alpha_k + 42\prod \alpha_i &= \\ = 2(\sum \alpha_i)^2(\sum \alpha_i\alpha_j) + (\sum \alpha_i\alpha_j)^2 + 12\prod \alpha_i \end{aligned}$$

$$\begin{aligned} \sum \alpha_i^3\alpha_j^2 + 3\sum \alpha_i^3\alpha_j\alpha_k + 7\sum \alpha_i^2\alpha_j^2\alpha_k + 15(\prod \alpha_i)(\sum \alpha_i) &= \\ = (\sum \alpha_i)(\sum \alpha_i\alpha_j)^2 + (\sum \alpha_i)^2(\sum \alpha_i\alpha_j\alpha_k) - 4(\sum \alpha_i)(\prod \alpha_i) \end{aligned}$$

$$\begin{aligned} \sum \alpha_i^3\alpha_j^2\alpha_k + 2\sum \alpha_i^2\alpha_j^2\alpha_k^2 + 2(\prod \alpha_i)(\sum \alpha_i^2) + 4(\prod \alpha_i)(\sum \alpha_i\alpha_j) &= \\ = (\sum \alpha_i)(\sum \alpha_i\alpha_j)(\sum \alpha_i\alpha_j\alpha_k) - (\sum \alpha_i)^2(\prod \alpha_i) - (\sum \alpha_i\alpha_j\alpha_k)^2 \end{aligned}$$

we find that

$$\begin{aligned}
c_1(\wedge^2\mathcal{E}) &= 3c_1\\
c_2(\wedge^2\mathcal{E}) &= 3c_1^2 + 2c_2\\
c_3(\wedge^2\mathcal{E}) &= c_1^3 + 4c_1c_2\\
c_4(\wedge^2\mathcal{E}) &= 2c_1c_2 + c_2^2 + 12c_4\\
c_5(\wedge^2\mathcal{E}) &= c_1c_2^2 + c_1^2c_3 - 4c_1c_4\\
c_6(\wedge^2\mathcal{E}) &= c_1c_2c_3 - c_1^2c_4 - c_3^2.
\end{aligned}$$

□

Exercise 1.107. **??** Let $\mathcal{E}$ be a vector bundle of rank 3. Express the Chern classes of $\operatorname{Sym}^2\mathcal{E}$ in terms of those of $\mathcal{E}$.

Solution to Exercise ??: In the same way as in the previous exercises, we split $\mathcal{E}$ in line bundles with first Chern class $\alpha_1, \alpha_2, \alpha_3$, and we get

$$c(\operatorname{Sym}^2\mathcal{E}) = (1+2\alpha_1)(1+2\alpha_2)(1+2\alpha_3)(1+\alpha_1+\alpha_2)(1+\alpha_1+\alpha_3)(1+\alpha_2+\alpha_3).$$

Collecting the first and the last three factors, and using Exercise **??**, we get

$$c(\operatorname{Sym}^2\mathcal{E}) = (1 + 2c_1 + 4c_2 + 8c_3)(1 + 2c_1 + c_1^2 + c_2 + c_1c_2 - c_3)$$

and expanding the product we get

$$\begin{aligned}
c_1(\operatorname{Sym}^2\mathcal{E}) &= 4c_1\\
c_2(\operatorname{Sym}^2\mathcal{E}) &= 5c_1^2 + 5c_2\\
c_3(\operatorname{Sym}^2\mathcal{E}) &= 2c_1^3 + 11c_1c_2 + 7c_3\\
c_4(\operatorname{Sym}^2\mathcal{E}) &= 6c_1^2c_2 + 14c_1c_3 + 4c_2^2\\
c_5(\operatorname{Sym}^2\mathcal{E}) &= 8c_1^2c_3 + 4c_1c_2^2 + 4c_2c_3\\
c_6(\operatorname{Sym}^2\mathcal{E}) &= 8c_1c_2c_3 - 8c_3^2.
\end{aligned}$$

□

Exercise 1.108. **??** Let $\mathcal{E}$ be a vector bundle of rank 2. Express the Chern classes of $\operatorname{Sym}^3\mathcal{E}$ in terms of those of $\mathcal{E}$.

Solution to Exercise ??: Splitting $\mathcal{E}$ into line bundles with first Chern classes α_1 and α_2, we get

$$c(\operatorname{Sym}^3\mathcal{E}) = (1 + 3\alpha_1)(1 + 3\alpha_2)(1 + 2\alpha_1 + \alpha_2)(1 + \alpha_1 + 2\alpha_2)$$

colleting the two first and the last two factors, we get

$$\begin{aligned}
c(\operatorname{Sym}^3\mathcal{E}) &= (1 + 3c_1 + 9c_2)(1 + c_1 + \alpha_1)(1 + c_1 + \alpha_2) =\\
&= (1 + 3c_1 + 9c_2)(1 + 3c_1 + 2c_1^2 + c_2) =\\
&= 1 + 6c_1 + (11c_1^2 + 10c_2) + (6c_1^3 + 30c_1c_2) + (18c_1^2c_2 + 9c_2^2)
\end{aligned}$$

we get all Chern classes of $\operatorname{Sym}^3\mathcal{E}$. □

Exercise 1.109. **??** Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of rank 2. Express the Chern classes of the tensor product $\mathcal{E} \otimes \mathcal{F}$ in terms of those of $\mathcal{E}$ and $\mathcal{F}$.

Solution to Exercise ??: Let us split $\mathcal{E}$ and $\mathcal{F}$ in line bundles with first Chern classes respectively α_1, α_2 and β_1, β_2. Let us call denote by (respectively) c_1, c_2 and d_1, d_2 the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. We then have

$$c(\mathcal{E} \otimes \mathcal{F}) = (1 + \alpha_1 + \beta_1)(1 + \alpha_1 + \beta_2)(1 + \alpha_2 + \beta_1)(1 + \alpha_2 + \beta_2)$$

and multiplying out we get

$$\begin{aligned} c_1(\mathcal{E} \otimes \mathcal{F}) &= 2c_1 + 2d_1 \\ c_2(\mathcal{E} \otimes \mathcal{F}) &= c_1^2 + 2c_2 + d_1^2 + 2d_2 + 3c_1d_1 \\ c_3(\mathcal{E} \otimes \mathcal{F}) &= 2c_1c_2 + 2d_1d_2 + c_1^2d_1 + c_1d_1^2 + 2c_1d_2 + 2c_2d_1 \\ c_4(\mathcal{E} \otimes \mathcal{F}) &= c_2^2 + d_2^2 + c_1c_2d_1 + c_1d_1d_2 + c_1^2d_2 + c_2d_1^2 - 2c_2d_2. \end{aligned}$$

□

Exercise 1.110. **??** Just to get a sense of how rapidly this gets complicated: do the preceding exercise for a pair of vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks 2 and 3.

Solution to Exercise ??: Let us use a slightly different idea: let us split only $\mathcal{E}$ in two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ with first Chern classes α_1 and α_2; we have now that the tensor product is the direct sum of $\mathcal{L}_1 \otimes \mathcal{F}$ and $\mathcal{L}_2 \otimes \mathcal{F}$, and we can use the formulas for the tensor product with a line bundle and then Whitney's formula. So, we have

$$\begin{aligned} c(\mathcal{L}_1 \otimes \mathcal{F}) &= \sum_l d_l(1 + \alpha_i)^{3-l} = \\ &= (1 + \alpha_i)^3 + d_1(1 + \alpha_i)^2 + d_2(1 + \alpha_i) + d_3 \end{aligned}$$

and after some manipulations, we get

$$\begin{aligned} c(\mathcal{E} \otimes \mathcal{F}) = {} & (1 + c_1 + c_2)^3 + d_1(1 + c_1 + c_2)^2(2 + c_1) + \\ & + d_2(1 + c_1 + c_2)(2 + 2c_1 + c_1^2 - 2c_2) + \\ & + d_3(2 + 3c_1 + 3c_1^2 - 6c_2 + c_1^3 - 3c_1c_2) + \\ & + d_1^2(1 + c_1 + c_2)^2 + d_1d_2(1 + c_1 + c_2)(2 + c_1) + \\ & + d_1d_3(2 + 2c_1 + c_1^2 - 2c_2) + d_2^2(1 + c_1 + c_2) + d_2d_3(2 + c_1) + d_3^2 \end{aligned}$$

that we will not simplify any further. □

Exercise 1.111. **??** Apply the preceding exercise to find all the Chern classes of the tangent bundle $\mathcal{T}_G$ of the Grassmannian $G = G(2, 4)$

Solution to Exercise ??: We have $\mathcal{T}_G = \mathcal{S}^* \otimes \mathcal{Q}$, and we know the Chern classes of $\mathcal{S}^*$ and $\mathcal{Q}$, that are

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1}$$
$$c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2$$

and now we can use Exercise **??** to find

$$c_1(\mathcal{T}_G) = 4\sigma_1$$
$$c_2(\mathcal{T}_G) = 7\sigma_2 + 7\sigma_{1,1}$$
$$c_3(\mathcal{T}_G) = 12\sigma_{2,1}$$
$$c_4(\mathcal{T}_G) = 6\sigma_{2,2}$$

that we can verify with Proposition **??** (about c_1) and Theorem **??** (about c_4). □

Exercise 1.112. **??** Find all the Chern classes of the tangent bundle $\mathcal{T}_Q$ of a quadric hypersurface $Q \subset \mathbb{P}^5$. Check that your answer agrees with your answer to the last exercise!

Solution to Exercise ??: We can use the formula in Section **??** that gives us

$$c(\mathcal{T}_Q) = \frac{(1+\zeta_Q)^6}{(1+2\zeta_Q)} = (1+\zeta_Q)^6(1 - 2\zeta_Q + 4\zeta_Q^2 - 8\zeta_Q^3 + 16\zeta_Q^4) =$$
$$= 1 + 4\zeta_Q + 7\zeta_Q^2 + 6\zeta_Q^3 + 3\zeta_Q^4$$

and the result is exactly the same as the previous exercise we realize that $\zeta_Q = \sigma_1$ because they both are then hyperplane section under the Plücker embedding. □

Exercise 1.113. **??** Calculate the Chern classes of the tangent bundle to a product $\mathbb{P}^n \times \mathbb{P}^m$ of projective spaces

Solution to Exercise ??: It is immediate to see that the tangent bundle of $\mathbb{P}^n \times \mathbb{P}^m$ splits as direct sum of the two tangent bundles $\mathcal{T}_{\mathbb{P}^n}$ and $\mathcal{T}_{\mathbb{P}^m}$; then, following the notation of Theorem **??**, Whitney's formula gives us

$$\mathcal{T}_{\mathbb{P}^n \otimes \mathbb{P}^m} = (1+\alpha)^{n+1}(1+\beta)^{m+1} =$$
$$= 1 + (n+1)\alpha + (m+1)\beta + \binom{n+1}{2}\alpha^2 + (n+1)(m+1)\alpha\beta + \dots.$$

□

Exercise 1.114. **??** Find the Euler characteristic of a smooth hypersurface of bidegree (a, b) in $\mathbb{P}^m \times \mathbb{P}^n$.

Solution to Exercise ??: Let us use Theorem **??** on the hypersurface X; we have

$$c(\mathcal{T}_X) = \frac{c(\mathcal{T}_{\mathbb{P}^n \otimes \mathbb{P}^m})}{(1 + a\alpha + b\beta)} =$$

$$= \left(\sum_{i,j} \binom{n+1}{i}\binom{m+1}{j}\alpha^i \beta^j\right)\left(\sum_{k,h}(-1)^{k+h}\binom{k+h}{k}a^k b^h \alpha^k \beta^h\right)$$

and the degree of the top Chern class is just the coefficient in this sum of $\alpha^n \beta^m$, so is obtained as

$$\chi_{\mathrm{top}}(X) = \sum_{k,h} \binom{n+1}{k+1}\binom{m+1}{h+1} \cdot (-1)^{k+h}\binom{k+h}{k}a^k b^h$$

that we can check for instance for $n = m = 1$, where we get $\chi_{\mathrm{top}}(X) = -2a-2b+2ab$, that agrees with the formula $g = (a-1)(b-1)$ of the genus of an (a, b) curve on $\mathbb{P}^1 \times \mathbb{P}^1$, and the fact that the Euler characteristic of a genus g curve is $\chi = 2-2g$. □

Exercise 1.115. ?? Using the Whitney formula, show that for $n \geq 2$ the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of projective space is not a direct sum of line bundles.

Solution to Exercise ??: If $\mathcal{T}_{\mathbb{P}^n}$ splitted as sum of line bundles, then the polynomial $p(x) = (1+x)^{n+1} - x^{n+1}$ would split in linear terms of the kind $(1 + ax)$ where a is an integer; in particular, $p(x)$ would only have rational roots. But over the complex numbers, $p(x)$ splits as

$$p(x) = \prod_{i=1}^{n}(1 + x - \zeta^i x)$$

where ζ_{n+1} is a primitive $(n+1)$th root of 1; so, unless $n + 1 = 2$ (so that $\mathcal{T}_{\mathbb{P}^1}$ is of course a line bundle), $p(x)$ has complex roots, hence it cannot split in linear factors with rational coefficients. Using the knowledge on the global sections of $\mathcal{T}_{\mathbb{P}^n}$ (coming from the Euler sequence) it is possible to prove that more in general $\mathcal{T}_{\mathbb{P}^n}$ is not a direct sum of subbundles of any positive rank. □

Exercise 1.116. ?? Find the Betti numbers of the smooth intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$, and of the intersection of three quadrics in $\mathbb{P}^5$. (Both of these are examples of *K3 surfaces*, which are diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$.)

Solution to Exercise ??: For what we have seen in Section **??**, we only need to find the middle Betti number, in this case h^2 because we are dealing with surfaces. In the first case, using Whitney's formula, we get

$$c(\mathcal{T}_S) = \frac{(1+\zeta)^5}{(1+2\zeta)(1+3\zeta)} = (1 + 5\zeta + 10\zeta^2)(1 - 5\zeta + 19\zeta^2) = 1 + 4\zeta^2$$

so $\chi_{top}(S) = deg(4\zeta^2) = 24$ because S is a surface of degree 6. So, we need to have $h^2 = 22$, so that we have

$$h^0 = 1 \quad h^1 = 0 \quad h^2 = 22 \quad h^3 = 0 \quad h^4 = 1.$$

For the latter, again Whitney's formula gives us

$$c(\mathcal{T}_S) = \frac{(1+\zeta)^6}{(1+2\zeta)^3} = (1 + 6\zeta + 15\zeta^2)(1 - 6\zeta + 24\zeta^2) = 1 + 3\zeta^2$$

and again we find $\chi_{top}(S) = deg(3\zeta^2) = 24$ because the surface is degree 8. Hence, the Betti numbers are the same as before. □

Exercise 1.117. ?? Find the Betti numbers of the smooth intersection of two quadrics in $\mathbb{P}^5$. This is the famous *quadric line complex*, about which you can read more in [GH], Chapter 6.

Solution to Exercise ??: As in the previous exercise, we can find

$$\begin{aligned} c(\mathcal{T}_X) &= \frac{(1+\zeta)^6}{(1+2\zeta)^2} = (1 + 6\zeta + 15\zeta^2 + 20\zeta^3)(1 - 4\zeta + 12\zeta^2 - 32\zeta^3) = \\ &= 1 + 2\zeta + 3\zeta^2 + 0 \cdot \zeta^3 \end{aligned}$$

so that $\chi_{top}(X) = 0$; from Lefschetz hyperplane theorem **ref** for complete intersections, we know that $h^0 = h^2 = h^4 = h^6 = 1$ and that $h^1 = h^5 = 0$; this gives us $h^3 = 4$. □

Exercise 1.118. ?? Show that the cohomology groups of a smooth quadric threefold $Q \subset \mathbb{P}^4$ are isomorphic to those of $\mathbb{P}^3$ ($\mathbb{Z}$ in even dimensions, 0 in odd), but its cohomology ring is different (the square of the generator of $H^2(Q,\mathbb{Z})$ is twice the generator of $H^4(Q,\mathbb{Z})$). (This is a useful example of the fact that two compact, oriented manifolds can have the same cohomology groups but different cohomology rings, if you're ever teaching a course in algebraic topology.)

Solution to Exercise ??: From Table **??** and Lefschetz theorem again, we have that Betti numbers of Q are the same of those of $\mathbb{P}^3$. Consider now the generator of $H^2(Q,\mathbb{Z})$; from Lefschetz theorem again, this comes from the generator of $H^2(\mathbb{P}^4,\mathbb{Z})$, so it is the hyperplane section; hence, is also the generator of $A^1(Q)$. Considering its self intersection, we get a curve of degree 2 in $\mathbb{P}^4$, that is a conic plane curve; this will also be the square of the cohomology class, because when intersection is transverse cup product is the same as intersection product. But now, Q also contains lines (from an incidence correspondence, it is easy to see that contains a 3 dimensional family of them), and the class of one of these lines has to be the generator of $A^2(Q)$; it is a generator of $H^2(Q, \mathbb{Z}) = \mathbb{Z}$ too, because it is indivisible (this can be seen from the fact that the intersection with the hyperplane section class is 1); now, the class of the conic above is clearly twice this generator, so we proved the claim; in particular, the Chow (and cohomology) rings of Q and $\mathbb{P}^3$ are isomorphic as additive groups, but not as rings. □

Exercise 1.119. ?? Let $S \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees d and e, and let $Y \subset \mathbb{P}^4$ be any hypersurface of degree f containing S. Show that if f is not equal to either d or e, then Y is necessarily singular.

(Hint: Assume Y is smooth, and apply the Whitney formula to the sequence

$$0 \to \mathcal{N}_{S/Y} \to \mathcal{N}_{S/\mathbb{P}^4} \to \mathcal{N}_{Y/\mathbb{P}^4}|_S \to 0$$

to arrive at a contradiction.)

Solution to Exercise ??: If Y is smooth, then $\mathcal{N}_{S/Y}$ is a line bundle on S. Consider the exact sequence in the hint; remember that $\mathcal{N}_{Y/\mathbb{P}^4}|_S \cong \mathcal{O}_{\mathbb{P}^4}(f)|_S$ and $\mathcal{N}_{S/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(d)|_S \oplus \mathcal{O}_{\mathbb{P}^4}(e)|_S$. Using Whitney's formula, we can find out the Chern classes of $\mathcal{N}_{S/Y}$, that means,

$$c(\mathcal{N}_{S/Y}) = \frac{(1 + d\zeta)(1 + e\zeta)}{1 + f\zeta} = 1 + (d + e - f)\zeta + (d - f)(e - f)\zeta^2.$$

In order for it to be the total Chern class of a line bundle, we need the second Chern class to be 0, that means, $f = d$ or $f = e$. Otherwise, Y is singular; note that this also prove that Y is singular *along* S, and the quantity $(d - f)(e - f)$ gives quantitative information about the singularities of Y (through the analysis of the torsion of the normal sheaf $\mathcal{N}_{S/Y}$). □

1.6 Chapter 6

Exercise 1.120. ?? Let $\mathbb{P}^N$ be the space of all surfaces of degree d in $\mathbb{P}^3$. Show that the set of surfaces of degree d containing two or more lines is a subvariety of codimension $2d - 6$ in $\mathbb{P}^N$. (Thus a general surface $X \subset \mathbb{P}^3$ of degree $d \geq 4$ containing a line contains only one line, and no surface in a general pencil of surfaces of degree d will contain more than one line.)

Solution to Exercise ??: Consider the incidence correspondence in $\mathbb{P}^N \times \mathbb{G}(1,3) \times \mathbb{G}(1,3)$ given by

$$\Phi = \{(X, L, M) \mid L \cup M \subset X\},$$

and suppose we know that it has dimension $N - 2d + 6$; then, to show that the image via the projection onto X has the same dimension, we need to show that is generically finite, so it is sufficient to show that it exists a degree d surface containing finitely many lines. For instance, we can consider the Fermat surface with equation $x_0^d + x_1^d + x_2^d + x_3^d = 0$, that indeed contains only finitely many lines (the check of this fact is left to the reader). So, we need to look at the dimension of the whole incidence correspondence Φ; let us now consider the fibers of the projection onto $\mathbb{G}(1,3) \times \mathbb{G}(1,3)$; that means, understanding the dimension of degree d polynomials vanishing on two given lines (L, M). This is the same as finding the dimension of the kernel of

$$r_{L,M} : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(L, \mathcal{O}_L(d)) \times H^0(M, \mathcal{O}_m(d)).$$

Now, if $r_{L,M}$ is surjective, then the kernel has dimension $N+1-2(d+1) = N-2d-1$, so taking the projectivization we would get $N-2d-2$-dimensional fiber over (L, M). Now, consider the case in which L and M are skew lines (the general case), and let's see that $r_{L,M}$ is surjective; we can then set coordinates in such a way x_0, x_1 are coordinates on L and vanish on M, and viceversa for x_2, x_3: now, considering any two degree d polynomials $f(x_0, x_1)$ and $g(x_2, x_3)$, they are in the image of the polynomial

$$f(x_0, x_1) + g(x_2, x_3) \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$$

so that $r_{L,M}$ is surjective and the fiber over (L, M) is $N-2d-2$-dimensional. If L, M meet in one point p, a similar argument shows that $r_{L,M}$ has a one dimensional cokernel (due to the fact that the two polynomials f and g have to be chosen to have the same value at p), so that the fiber over (L, M) is $N-2d-1$-dimensional (we do not consider the case $L = M$, because in its fiber we would find all surfaces containing one line). So, adding everything up, we get that Φ is $N-2d+6$-dimensional, that completes the proof. □

Exercise 1.121. **??** Show that the expected number of lines on a hypersurface of degree $2n-3$ in $\mathbb{P}^n$ (that is, the degree of $c_{2n-2}(\mathrm{Sym}^{2n-3}\mathcal{S}^*) \in A(\mathbb{G}(1,n))$) is always positive, and deduce that *every hypersurface of degree* $2n-3$ *in* $\mathbb{P}^n$ *must contain a line*. (This is just a special case of Corollary **??**; the idea here is to do it without a tangent space calculation.)

Solution to Exercise ??: Using the splitting principle, we have

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1} = (1+\alpha)(1+\beta)$$

so that we obtain

$$c(\mathrm{Sym}^{2n-3}\mathcal{S}^*) = (1+(2n-3)\alpha)(1+(2n-4)\alpha+\beta)\dots(1+(2n-3)\beta)$$

and hence

$$c_{15}(\mathrm{Sym}^{2n-3}\mathcal{S}^*) = (2n-3)\alpha((2n-4)\alpha+\beta)\dots(2n-3)\beta$$

collecting terms two by two starting from the two extremal, we get, as i goes from 0 to

$n-2$,

$$\begin{aligned}&((2n-3-i)\alpha + i\beta)(i\alpha + (2n-3-i)\beta)\\ =&i(2n-3-i)(\alpha^2+\beta^2) + (i^2 + (2n-3-i)^2)\alpha\beta =\\ =&i(2n-3-i)(\sigma_2 - \sigma_{1,1}) + (i^2 + (2n-3-i)^2)\sigma_{1,1} =\\ =&i(2n-3-i)\sigma_2 + (i^2 - i(2n-3-i) + (2n-3-i)^2)\sigma_{1,1}\end{aligned}$$

that is, a positive linear combination of σ_2 and $\sigma_{1,1}$; the product of all such elements for i going from 0 to $n-2$ is then going to be a positive multiple of $\sigma_{n-1,n-1}$. □

Exercise 1.122. **??** Let $X \subset \mathbb{P}^4$ be a general quartic threefold. By Theorem **??**, X will contain a one-parameter family of lines. Find the class in $A(\mathbb{G}(1,4))$ of the Fano scheme $F_1(X)$, and the degree of the surface $Y \subset \mathbb{P}^4$ swept out by these lines.

Solution to Exercise ??: From Theorem **??**, we have that $F_1(X)$ is one dimensional and reduced; so, its class will be exactly $c_5(\mathrm{Sym}^4 \mathcal{S}^*)$. In order to find this, we split $\mathcal{S}^*$ in two (virtual) line bundles with Chern classes α and β, and we get

$$\begin{aligned}[F_1(X)] = c_5(\mathrm{Sym}^4 \mathcal{S}^*) &= 4\alpha(3\alpha+\beta)(2\alpha+2\beta)(\alpha+3\beta)4\beta\\ &= 32\alpha\beta(\alpha+\beta)(3(\alpha+\beta)^2 + 4\alpha\beta) =\\ &= 32\sigma_{1,1}\sigma_1(3\sigma_1^2 + 4\sigma_{1,1}) = 320\sigma_{3,2}.\end{aligned}$$

Using Exercise **??**, the degree of the surface swept out is $[F_1(X)]\cdot\sigma_1 = 320$. □

Exercise 1.123. **??** Find the class of the scheme $F_2(Q) \subset \mathbb{G}(2,5)$ of 2-planes on a quadric $Q \subset \mathbb{P}^5$. (Do the problem first, then compare your answer to the result in Proposition **??**.)

Solution to Exercise ??: As in the previous exercises, we need to find the class $c_6(\mathrm{Sym}^2 \mathcal{S}^*)$ Splitting $\mathcal{S}^*$ in three (virtual) line bundles with Chern classes α, β and γ, we get

$$\begin{aligned}c_6(\mathrm{Sym}^2 \mathcal{S}^*) &= 2\alpha\cdot 2\beta\cdot 2\gamma(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma) =\\ &= 8\sigma_{1,1,1}(\sigma_1\sigma_{1,1} - \sigma_{1,1,1}) = 8\sigma_{3,2,1}\end{aligned}$$

that in fact agrees with Proposition **??**. □

Exercise 1.124. **??** Find the expected number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$, that is, the degree of $c_{15}(\mathrm{Sym}^4 \mathcal{S}^*) \in A(\mathbb{G}(2,7))$.

Solution to Exercise ??: After a very long calculation (that we will not present here), we get

$$c_{15}(\mathrm{Sym}^4 \mathcal{S}^*) = 3297280\sigma_{5,5,5}$$

so we get that the expected number of 2-planes on a general quartic sixfold 3297280.

We will extend Theorem **??** to this case, and then prove that a general quartic sixfold contains exactly that many 2-planes, in Exercises **??**-**??**. □

Exercise 1.125. **??** We can also use the calculation carried out in this chapter to count lines on complete intersections $X = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$, simply by finding the classes of the schemes $F_1(Z_i)$ of lines on the hypersurfaces Z_i and multiplying them in $A(\mathbb{G}(1,n))$. Do this to find the number of lines on the intersection $X = Y_1 \cap Y_2 \subset \mathbb{P}^5$ of two general cubic hypersurfaces in $\mathbb{P}^5$.

Solution to Exercise ??: To find the lines in a cubic fourfold, we split $\mathcal{S}^*$ on $\mathbb{G}(1,5)$ using the virtual classes α and β; we then need to find

$$\begin{aligned} c_4(\operatorname{Sym}^3 \mathcal{S}^*) &= 3\alpha(2\alpha+\beta)(\alpha+2\beta)3\beta = 9\alpha\beta(2(\alpha+\beta)^2+\alpha\beta) = \\ &= 9\sigma_{1,1}(2\sigma_1^2+\sigma_{1,1}) = 18\sigma_{3,1} + 27\sigma_{2,2}. \end{aligned}$$

So its square in $A^*(\mathbb{G}(1,5))$ is $(18^2+27^2)\sigma_{4,4} = 1053\sigma_{4,4}$, so the degree is 1053. An argument such as in Theorem **??** will also prove that for a general complete intersection of this kind the Fano scheme is reduced, so X contains exactly 1053 lines. □

Exercise 1.126. **??** Find the Chern class $c_3(\operatorname{Sym}^3 \mathcal{S}^*) \in A^3(\mathbb{G}(1,3))$. Why is this the degree of the curve of lines on the cubic surfaces in a pencil? Note that this computation does not use the incidence correspondence Φ.

Solution to Exercise ??: To find this class, splitting $\mathcal{S}^*$ again using the virtual classes α and β, we get

$$\begin{aligned} c(\operatorname{Sym}^3 \mathcal{S}^*) &= (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta) = \\ &= 1 + 6(\alpha+\beta) + 11(\alpha+\beta)^2 + 10\alpha\beta + 6(\alpha+\beta)^3 + \\ &+ 30\alpha\beta(\alpha+\beta) + 9\alpha\beta(2(\alpha+\beta)^2 + \alpha\beta = \\ &= 1 + 6\sigma_1 + 11\sigma_2 + 21\sigma_{1,1} + 42\sigma_{2,1} + 27\sigma_{2,2}. \end{aligned}$$

So, we get $c_3(\operatorname{Sym}^3 \mathcal{S}^*) = 42\sigma_{2,1}$ that means, a curve of degree 42. For the second part of the exercise, let us remember Theorem **??**: for a rank 4 vector bundle, as $c_3(\operatorname{Sym}^3 \mathcal{S}^*)$ is, the class c_3 is the locus τ_0, τ_1 where two sections become dependent; that means, where a linear combination of τ_0 and τ_1 vanishes; but now, if τ_0 and τ_1 come from two cubic polynomials f and g, the linear combinations will be all cubics is the pencil generated by f and g; so, c_3 is exactly the class of lines belonging to cubics in a pencil. □

Exercise 1.127. **??** Let $\{X_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^2}$ be a general net of cubic surfaces in $\mathbb{P}^3$.

(a) Let $p \in \mathbb{P}^3$ be a general point. How many lines lying on some member X_t of the net pass through p?

(b) Let $H \subset \mathbb{P}^3$ be a general plane. How many lines lying on some member X_t of the net lie in H?

Compare your answer to the second half of this question to the calculation in Chapter **??** of the degree of the locus of reducible plane cubics!

Solution to Exercise ??: In the same fashion as in the previous exercise, the set of lines lying in one of the cubics of the net is $c_2(\mathrm{Sym}^3 \mathcal{S}^*) = 11\sigma_2 + 21\sigma_{1,1}$. Asking how many lines of this locus pass through a point p is the same as finding its intersection with a general $\Sigma_2(p)$: the answer is then $c_2(\mathrm{Sym}^3 \mathcal{S}^*) \cdot \sigma_2 = 11\sigma_{2,2}$ so the answer is 11. For the second part, we need to intersect with $\Sigma_{1,1}(H)$, so that we get 21. For the last observation, let us put ourselves in the point of view of the plane: restricting the net to H, we get a net of plane degree 3 curves, and we are asking how many of them contain a line, that means, are reducible. But as we found out in Section **??**, this is locus has codimension two and degree 21, so 21 will be the points of intersection with a general net of cubics. □

Exercise 1.128. **??** Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 3$. Show that if $F_1(X)$ is positive-dimensional, then either X is a cone or X has a positive-dimensional singular locus.

Solution to Exercise ??: As we have seen in Proposition **??**, the only possibility for $F_1(X)$ to be 2-dimensional is if X is a plane; so, let us concentrate on the case of $F_1(X)$ being a curve in $\mathbb{G}(1,3)$. Let now $L \subset X$ be a line; if X was smooth along L, we would have

$$T_L F_1(X) = H^0(\mathcal{N}_{L/X}) = H^0(\mathcal{O}_L(2-d)) = 0$$

bu this is impossible because $T_L F_1(X)$ is (at least) one dimensional in this case; this proves that every L in $F_1(X)$ meets X^{sing}. Now, if we collect all these singular points on this line, either we get a single point (so that all lines meet in a single point, and X is a cone) or a positive dimensional singular locus. □

Exercise 1.129. **??** Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold, and

$$\{S_t = X \cap H_t\}_{t \in \mathbb{P}^1}$$

a general pencil of hyperplane sections of X. What is the degree of the surface swept out by the lines on the surfaces S_t, and what is the genus of the curve parametrizing them?

Solution to Exercise ??: The class of all lines in a cubic threefold is given again by $c_4(\mathrm{Sym}^3 \mathcal{S}^*)$, that in $\mathbb{G}(1,4)$ is the class $18\sigma_{3,1} + 27\sigma_{2,2}$; the ones that also belong to one of the hyperplanes of the pencil, are exactly those meeting the $\mathbb{P}^2$ that is the base locus of the pencil; so, we just need to intersect with σ_1, and we get $45\sigma_{3,2}$: we get then a curve C of degree 45. To find the genus, we can consider the map $C \to \mathbb{P}^1$, obtained sending every line to the element of the pencil it belongs; this map is well defined, because we can choose the pencil general enough having no line of X in the base $\mathbb{P}^2$. Now, wherever the hyperplane section is smooth (i. e. where the hyperplane section is a smooth cubic surface) this map has degree 27; considering a general pencil of hyperplanes, then, we can obtain singular fibers to be only cubic surfaces with one node, so that fibers are composed by exactly 21 lines (in particular, we have 6 ramification points of order 2); the number of such situation is the same as the degree of the dual hypersurface to X, that from what we did in Section **??** is 24. So, we can use Riemann-Hurwitz formula, to get

$$2 \cdot 27 + 2g - 2 = 24 \cdot 6$$

that means, $g = 46$. One should prove that the curve is in fact connected and nonsingular; otherwise, we only proved that the *arithmetic genus* of C is 46. □

Exercise 1.130. **??** Prove Theorem **??** using the methods of Section **??**, that is, by writing the local equations of $F_k(X) \subset \mathbb{G}(k, n)$

Solution to Exercise ??: Let us consider projective coordinates $x_0, \ldots, x_n$ such that the plane L is the plane $x_{k+1} = \ldots = x_n = 0$. We can write as usual the defining polynomial of X as

$$f(x) = \sum_{i=k+1}^{n} x_i f_i(x_0, \ldots, x_k) + g$$

where $g \in (x_{k+1}, \ldots, x_n)^2$. Note that as we do Section **??**, we can see $x_0, \ldots, x_k$ as local coordinates for L, and $x_{k+1}, \ldots, x_n$ as linear functions of those, i. e. as sections of $\mathcal{O}_L(1)$; using the identification $T_L\mathbb{G} \cong H^0(\mathcal{N}_{L/\mathbb{P}^n}) = H^0(\mathcal{O}_L(1)^{n-k})$, we get that a vector in $T_L\mathbb{G}$ is exactly one choice of $n - k$ sections $x_{k+1}, \ldots, x_n$ of $\mathcal{O}_L(1)$. To have this vector also lying in $T_L F_k(X)$, using again Section **??**, we need the further condition that

$$\sum_{i=k+1}^{n} x_i f_i(x_0, \ldots, x_k) \equiv 0$$

as polynomials of degree d in $x_0, \ldots, x_k$. Let us now consider the exact sequence for normal bundles for L, X and $\mathbb{P}^n$, and its long exact sequence in cohomology:

$$0 \to H^0(\mathcal{N}_{L/X}) \to H^0(\mathcal{N}_{L/\mathbb{P}^n}) \to H^0(\mathcal{N}_{X/\mathbb{P}^n})$$

$$0 \to H^0(\mathcal{N}_{L/X}) \to H^0(\mathcal{O}_L(1)^{n-k}) \xrightarrow{h} H^0(\mathcal{O}_L(d)).$$

We have now that the map h is just the multiplication (the 'dot product") with the degree $d - 1$ polynomials $f_{k+1}, \ldots, f_n$; so, kernel of h, from above, is exactly $T_L F_k(X)$; but it is also $H^0(\mathcal{N}_{L/X})$, because that is an exact sequence. □

Exercise 1.131. **??** Extending the results of Section **??**, suppose that X is a general cubic surface having two ordinary double points $p, q \in X$. Describe the scheme structure of $F_1(X)$ at the point corresponding to the line $L = \overline{p\,q}$, and in particular determine the multiplicity of $F_1(X)$ at L.

Solution to Exercise ??: Let us proceed in the same way as in Section **??**. Let us again assume that $L : X_2 = x_3 = 0$ (with the neighborhood in $\mathbb{G}(1,3)$ parametrized in the same way by a_2, a_3, b_2, b_3), that p is the point $(1,0,0,0)$, and that the tangent cone of X at p is $x_1x_3 + x_2^2 = 0$; we get then the same equation

$$g(x) = x_0x_1x_3 + x_0x_2^2 + \alpha x_1^2x_2 + \beta x_1^2x_3 + \gamma x_1x_2^2 + \delta x_1x_2x_3 + \epsilon x_1x_3^2 + k,$$

and now let us impose the further condition to be singular at q, that we can say is the point $(0,1,0,0)$: we obtain the conditions $\alpha = 0$ and $\beta = 0$ (then the surface is general so we can assume $\gamma \neq 0$; in particular, this condition is the same as the tangent cone to q to be a smooth conic rather than two lines). Now, we can write down the local equations for $F_1(X)$, as in Section **??**: we get

$$\begin{cases} a_2^2 = 0 \\ a_3 + 2a_2b_2 + \gamma a_2^2 + \delta a_2a_3 + \epsilon a_3^2 = 0 \\ b_3 + b_2^2 + 2\gamma a_2b_2 + \delta(a_2b_3 + a_3b_2) + 2\epsilon a_3b_3 = 0 \\ \gamma b_2^2 + \delta b_2b_3 + \epsilon b_3^2 = 0 \end{cases}$$

Considering the first order terms we go on the plane $a_3 = b_3 = 0$, and on this plane the two other equation cut the ideal $(a_2^2, \gamma b_2^2)$ so the intersection is an ideal of degree 4. This proves that the line $L = \overline{p\,q}$ is a point of multiplicity 4 of $F_1(X)$. □

Exercise 1.132. **??** Now let $X \subset \mathbb{P}^3$ be a cubic surface and $p, q \in X$ isolated singular points of X; let $L = \overline{p\,q}$. Show that L is an isolated point of $F_1(X)$, and that the multiplicity

$$\text{mult}_L\, F_1(X) \geq 4$$

Solution to Exercise ??: Note that by Exercise **??**, if L is not an isolated point of $F_1(X)$, then X is a cone over a point, but than it couldn't possibly have two isolated singular points, or it is singular along a curve; thit is impossible too, because in Exercise **??** we also found out that this singular locus meet all the lines in $F_1(X)$, so that p or q wouldn't be isolated singularities anymore. The only thing to prove then is that $\text{mult}_L\, F_1(X) \geq 4$; we will see that this situation arises as degeneration of the situation of the previous exercise, that will prove the claim.

This exercise is only slightly different from the previous one; the only difference is that here we are not assuming anymore that p and q are ordinary double points of X; in particular, we cannot use anymore the equation $g(x)$ above, because it assumes that p

has tangent cone $x_1x_3 + x_2^2$; something we can assume though, is that the tangent cone is of the form $\lambda x_1x_3 + \mu x_2^2 + \eta x_2x_3 + \xi x_3^2$, so that the equation becomes

$$g(x) = \lambda x_0x_1x_3 + \mu x_0x_2^2 + \eta x_0x_2x_3 + \xi x_0x_3^2 + \gamma x_1x_2^2 + \delta x_1x_2x_3 + \epsilon x_1x_3^2 + k$$

and now, given the fact that when λ, μ, η, ξ and γ are general we are in the situation of the previous exercise, we can in fact obtain X as limit of surfaces with ordinary double points at p at q, all containing L, so that the multiplicity of L in the Fano scheme has to be at least the same, that is, $\text{mult}_L F_1(X) \geq 4$. □

Exercise 1.133. ?? Let $X \subset \mathbb{P}^3$ be a cubic surface and $p_1, \ldots, p_\delta$ isolated singular points of X. Show that no three of the points p_i are collinear.

Solution to Exercise ??: Let us work in coordinates: using notations from the previous exercises, the equation of a surface that is singular at $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ (and hence contains the line joining them) is

$$g(x) = \lambda x_0x_1x_3 + \mu x_0x_2^2 + \eta x_0x_2x_3 + \xi x_0x_3^2 + \gamma x_1x_2^2 + \delta x_1x_2x_3 + \epsilon x_1x_3^2 + k.$$

Let us now impose it to be singular along a third point on the line (without loss of generality, we can assume it to be $(1, 1, 0, 0)$); the condition is $\lambda = 0$, that means that the polynomial is contained in the ideal $(x_2, x_3)^2$, so that the surface is singular along the whole line. So, it cannot be that three of the isolated singularities are collinear points. □

Exercise 1.134. ?? Use the result of the preceding two exercises to deduce the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points.*

Solution to Exercise ??: Suppose X has only isolated singularities, and at most two; then, $F_1(X)$ has to be zero dimensional; hence, its degree has to be 27. Now, let us denote by $p_1, \ldots, p_\delta$ the singularities: we have that every line $\overline{p_ip_j}$ is an isolated point of $F_1(X)$, of multiplicity at least 4 by Exercise **??**. Now, doing the math we get

$$27 \geq \deg F_1(X) \geq \sum \text{mult}_{\overline{p_ip_j}} F_1(X) \geq 4\binom{\delta}{2}$$

that gives us $\delta \leq 4$. One example of such cubics surface is the one given by the equation $\sum_{i \neq j} x_i^2x_j^2 = 0$. □

Exercise 1.135. ?? Using the methods of Section **??**, show that there exists a pair (X, Λ) with $X \subset \mathbb{P}^7$ a quartic hypersurface and $\Lambda \subset X$ a 2-plane such that Λ is an isolated, reduced point of $F_2(X)$.

Solution to Exercise ??: Let Λ be the plane $(x_3, \ldots, x_7)$, so we can parametrize a neighborhood of it as

$$A = \begin{pmatrix} 1 & 0 & 0 & a_{13} & \dots & a_{17} \\ 0 & 1 & 0 & a_{23} & \dots & a_{27} \\ 0 & 0 & 1 & a_{33} & \dots & a_{37} \end{pmatrix}.$$

Consider now the quartic X whose defining polynomial is the following:

$$g(x) = (x_0^3+x_1^3+x_2^3)x_3+(x_1^2+x_2^2)x_0x_4+(x_0^2+x_2^2)x_1x_5+(x_0^2+x_1^2)x_2x_6+x_0x_1x_2x_7$$

and let us prove that $F_1(X)$ has an isolated and reduced point at Λ; note that X contains the 4-plane (x_0, x_1, x_2), and fact is triple along it, so that $F_1(X)$ is going to be very wild away from Λ; in particular, it will have a component of (at least) dimension 6. If we call s, t, u projective coordinates on a 2-plane in the neighborhood of Λ, plugging in into $g(x)$ we get

$$g(s,t,u) = (s^3 + t^3 + u^3)(a_{13}s + a_{23}t + a_{33}u) + (st^2 + su^2)(a_{14}s + a_{24}t + a_{34}u) + \\ + (s^2t + tu^2)(a_{15}s + a_{25}t + a_{35}u) + (s^2u + t^2u)(a_{16}s + a_{26}t + a_{36}u) + stu(a_{17}s + a_{27}$$

and the equations for $F_1(X)$ obtained setting to zero the coefficients of that; at first, it is easy to see that these are only linear polynomials. Let us now use a “smart” process of elimination to the equations arising in this way; looking at coefficients of s^4, t^4, u^4, we immediately get that $a_{13} = a_{23} = a_{33} = 0$; looking at coefficients of $s^3t, s^3u, \dots, tu^3$, and using what we just learnt about a_{i3}, we get that also that

$$a_{24} = a_{34} = a_{15} = a_{35} = a_{16} = a_{26} = 0.$$

Looking at coefficients of s^2tu, st^2u, stu^2, we get $a_{17} = a_{27} = a_{37} = 0$; now, looking at the remaining three equations (for coefficients of s^2t^2, s^2u^2, t^2u^2) we get the equations

$$\begin{cases} a_{14} + a_{25} = 0 \\ a_{14} + a_{36} = 0 \\ a_{25} + a_{36} = 0 \end{cases}$$

whose only solution is when all three are zero. This proves that Λ is transverse intersection of hyperplanes, so is a reduced point. □

Exercise 1.136. **??** Using the result of Exercise **??**, show that the number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$ is the number calculated in Exercise **??** (that is, the Fano scheme $F_2(X)$ is reduced for X general).

Solution to Exercise ??: We can use the proof of Theorem **??** without any modification; when invoking Corollary **??**, about the existence of a smooth point in a single $F_1(X)$, we can instead use the previous exercise. □

Exercise 1.137. **??** To complete the proof of Proposition **??**, let $X \subset \mathbb{P}^3$ is a cubic surface with one ordinary double point $p = (1, 0, 0, 0)$, given as the zero locus of the cubic

$$F(Z_0, Z_1, Z_2, Z_3) = Z_0 A(Z_1, Z_2, Z_3) + B(Z_1, Z_2, Z_3)$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. If we write a line $L \subset X$ through p as the span $L = \overline{pq}$ with $q = (0, Z_1, Z_2, Z_3)$, show that X is smooth along $L \setminus \{p\}$ if and only if the zero loci of A and B intersect transversely at (Z_1, Z_2, Z_3).

Solution to Exercise ??: Let us suppose, without loss of generality, that the zero loci of A and B meet at $(1, 0, 0)$, and that the tangent line of the curve A at $(1, 0, 0)$ is $Z_3 = 0$. This means that we have

$$F(Z_0, Z_1, Z_2, Z_3) = Z_0 Z_1 Z_3 + \alpha Z_1^2 Z_2 + \beta Z_1^2 Z_3 + G(Z_0, Z_1, Z_2, Z_3)$$

where $G \in (Z_2, Z_3)^2$. Now, the condition of A and B to meet transversely is $\alpha \neq 0$ (because the tangent line to B at $(1, 0, 0)$ is $\alpha Z_2 + \beta Z_3 = 0$). But we also have that $\alpha = 0$ if and only if F is singular at $(-\beta, 1, 0, 0)$, so the claim is proved. □

Exercise 1.138. **??** Show that there exists a smooth quintic threefold $X \subset \mathbb{P}^4$ whose scheme $F_1(X)$ of lines contains an isolated point of multiplicity 2.

Solution to Exercise ??: Let us condier the quintic X defined by the equation

$$g(x) = x_0^4 x_2 + x_0^2 x_1 x_3 (x_1 + x_3) + x_1^4 x_4 + x_2^5 + x_3^5 + x_4^5 = 0$$

that is smooth along L. Then, one can prove using the tools as in Section **??** that $F_1(X)$ has an isolated point of multiplicity 2 at the line defined by the ideal (x_2, x_3, x_4). Now, this quintic is not necessarily smooth; but if we consider the linear system of quintics

$$V = \{f \,|\, f \in (x_2, x_3, x_4)^3 + (g)\}$$

so that all elements have the same behavior up to degree 3 as g along L, and so that for all of those the Fano scheme has multiplicity exactly 2 at L; this linear system separates points outside of L, so its base locus is exactly L; by Bertini's theorem, hence, the general element of V can be singular only along V; but g itself is smooth along L, so the general element of V willbe smooth everywhere. □

Exercise 1.139. **??** Let Φ be the incidence correspondence of triples consisting of a hypersurface $X \subset \mathbb{P}^n$ of degree $d = 2n - 3$, a line $L \subset X$ and a singular point p of X lying on L: that is,

$$\Phi = \{(X, L, p) \in \mathbb{P}^N \times \mathbb{G}(1, n) \times \mathbb{P}^n \mid p \in L \subset X \text{ and } p \in X_{\text{sing}}\}.$$

Show that Φ is irreducible.

Solution to Exercise ??: Let $\Psi \subset \mathbb{G}(1,n) \times \mathbb{P}^n$ be the set of couples (L, p) such that $p \in L$; then, the image of Φ by the projection onto $\mathbb{G}(1,n) \times \mathbb{P}^n$, is surjective onto Ψ, that is irreducible. Fibers of this map $\Phi \to \Psi$ are then just linear subspaces of $\mathbb{P}^N$, of codimension $n + d - 1$ (one can check this going in coordinates) so that fibers are irreducible, and hence Φ is too. □

Exercise 1.140. **??** Suppose F and G are two quartic polynomials on $\mathbb{P}^3$ and $\{X_t = V(F + tG)\}$ the pencil of quartics they generate; let σ_F and σ_G be the sections of $\mathrm{Sym}^4 \mathcal{S}^*$ corresponding to F and G. Let X_{t_0} be a member of the pencil containing a line $L \subset \mathbb{P}^3$.

(a) Find the condition on F and G that L is a reduced point of $V(\sigma_F \wedge \sigma_G) \subset \mathbb{G}(1,3)$.
(b) Show that this is equivalent to the condition that the point $(t_0, L) \in \mathbb{P}^1 \times \mathbb{G}(1,3)$ is a simple zero of the map $\mu^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \nu^* \mathrm{Sym}^d \mathcal{S}^*$.

Solution to Exercise ??: Let us pick coordinates for which $L = (x_2, x_3)$, and let us pick coordinates a, b, c, d for a neighborhood U of L in $\mathbb{G}(1,3)$ and projective coordinates s, t for lines in U; the equation of X_{t_0} will be $H := F + t_0 G = x_2 h_2(x_0, x_1) + x_3 h_3(x_0, x_1) + k$ where $k \in (x_2, x_3)^2$; then, both F and G (up to a constant factor) will contain terms only involving x_0, x_1, and we will call this polynomial $f_0(x_0, x_1)$ (its intersection with L gives the intersection of L with the base locus of the pencil). Now, we want to translate all of this to sections of vector bundles on $\mathbb{G}(1,3)$; so, we plug in s and t as in the previous exercises: note that monomials $s^4, \dots, t^4$ are a basis of a trivialization of $\mathrm{Sym}^4 \mathcal{S}^*$ on U; note also that we want to prove transversality, so we can forget about all terms of degree 2 in a, b, c, d; we then have

$$\sigma_F \wedge \sigma_G = \sigma_H \wedge \sigma_G = (as + bt)h_2(s,t) + (cs + dt)h_3(s,t) \wedge f_0(s,t) + k'$$

(k' is now in $(a, b, c, d)^2$) and we want this to be zero if and only if $a = b = c = d = 0$; in other words, we want the polynomials $sh_2, th_2, sh_3, th_3, f_0$ to be independent polynomials of degree 4 in s and t, that means, a basis. Another interpretation is the following; as we did in Exercise **??**, we can consider the exact sequence of normal bundles for L inside the surface S defined by H, and looking in cohonology we have

$$0 \to H^0(\mathcal{N}_{L/S}) \xrightarrow{u} H^0(\mathcal{N}_{L/\mathbb{P}^3}) \xrightarrow{p} H^0(\mathcal{N}_{S/\mathbb{P}^3}) \xrightarrow{q} H^1(\mathcal{N}_{L/S})$$

$$0 \to 0 = H^0(\mathcal{O}_L(-2)) \xrightarrow{u} H^0(\mathcal{O}_L(1)^{\oplus 2}) \xrightarrow{p} H^0(\mathcal{O}_L(4)) \xrightarrow{q} H^1(\mathcal{O}_L(-2))$$

and it is easy to see, still following Exercise **??**, that the condition we just expressed is the same as G not lying in the image of p, or, that is the same, not sent to zero by q in $H^1(\mathcal{N}_{L/S})$ (that is isomorphic to $H^0(\mathcal{O}_L)$ by Serre duality).

For the second point, let us pick the coordinate t for $\mathbb{P}^1$; choosing a parameter u for fibers of $\mathcal{O}_{\mathbb{P}^1}(-1)$, over a point (t, M) the map of vector bundles is just

$$u \mapsto u \cdot (F + tG)(s, t).$$

Using the fact that $F + tG = H + (t - t_0)G$, around the point (t_0, L) the map will be

$$u \to u \cdot ((as + bt)h_2(s,t) + (cs + dt)h_3(s,t) + (t - t_0)f_0(s,t) + k'')$$

(k'' is now in $(a, b, c, d, t - t_0)^2$) so that again the zero is simple if and only if the 5 polynomials are independent. □

Exercise 1.141. **??** Let $\Sigma \subset \mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$ containing a line. Interpret the condition of the preceding problem in terms of the geometry of the pencil $\mathcal{D}$ around the line L, and use this to answer two questions:

(a) What is the singular locus of Σ?
(b) What is the tangent hyperplane $\mathbb{T}_X\Sigma$ at a smooth point corresponding to a smooth quartic surface X containing a single line?

Solution to Exercise ??: Let H, S and L be as in the previous exercise; if H is a singular point of Σ, then for every G the condition above is not satisfied. From the polynomials point of view, to have $sh_2, th_2, sh_3, th_3, f_0$ never independent, we need to have sh_2, th_2, sh_3, th_3 linearly dependent, that means that h_2 and h_3 have (at least) two common roots, so that S is singular along L at at least two points. From the cohomology point of view, in the exact sequence

$$0 \to H^0(\mathcal{N}_{L/S}) \xrightarrow{u} H^0(\mathcal{N}_{L/\mathbb{P}^3}) \xrightarrow{p} H^0(\mathcal{N}_{S/\mathbb{P}^3}) \xrightarrow{q} H^1(\mathcal{N}_{L/S})$$

we need p to have a kernel, that means $H^0(\mathcal{N}_{L/S})$ to be nonzero; it is easy to show that if S is singular along L only at one point, then $H^0(\mathcal{N}_{L/S})$ is still zero, so we need at least two singular points. In conclusion, Σ^{sing} is the locus of quartics containing a line, and being singular in at least two points along it (note that unlike in the case of cubics, having two isolated singular points does not necessarily mean to contain the line joining them).

Let now H be a point of $\Sigma \setminus \Sigma^{sing}$, and let us find the tangent plane at H, meaning all G not satisfying the condition of the previous exercise. Note that it is clear (from both points of view) that this is going to be codimension 1 in $\mathbb{P}^n$; using cohomology, the condition for G is to lie in the image of

$$H^0(\mathcal{N}_{L/\mathbb{P}^3}) \xrightarrow{p} H^0(\mathcal{N}_{S/\mathbb{P}^3}).$$

Roughly, this means that a small deformation of H towards G, contains one of the lines obtained deforming L by a vector in $T_L\mathbb{G}(1,3)$ (if you think about it, this is exactly what we should expect from the tangent space to Σ to be!). Considering polynomials, we need f_0 to be in the linear span of sh_2, th_2, sh_3, th_3; a geometric interpretation of this is the following: for every vector $v \in T_L\mathbb{G}(1,3)$ (i. e. linear combination of those 4 polynomials), we have 4 points on the line, obtained as zero of the degree 4 polynomial, but also obtained intersecting the first order deformation of L by v with S; we want now the base locus of the family spanned by H and G to intersect L in a quadruple of points arising in this way, obtained from a vector in $T_L\mathbb{G}(1,3)$. □

The following two exercises give constructions of smooth hypersurfaces containing more than the expected families of lines.

Exercise 1.142. ?? Let $Z \subset \mathbb{P}^{n-2}$ be any smooth hypersurface. Show that the cone $\overline{pZ} \subset \mathbb{P}^{n-1}$ over Z in $\mathbb{P}^{n-1}$ is the hyperplane section of a smooth hypersurface $X \subset \mathbb{P}^n$, and hence that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Solution to Exercise ??: Let us pick coordinates $x_0, \dots, x_n$ for which $\mathbb{P}^{n-2}$ is the vanishing locus of x_0 and x_1, and $\mathbb{P}^{n-1}$ is the vanishing locus of x_0; let $f(x_2, \dots, x_n)$ be the polynomial defining Z; note that in $\mathbb{P}^{n-1}$ the equation of $\overline{pZ}$ is f again. Consider now the hypersurface $\overline{Z}$ defined by the polynomial

$$\overline{f}(x_0, \dots, x_n) = x_0^d + x_0 x_1^{d-1} + f(x_2, \dots, x_n)$$

whose hyperplane section $x_0 = 0$ is the cone $\overline{pZ}$. This hypersurface is smooth: in fact, from the fact that Z, the only way for the partial derivatives $\partial/\partial x_i$ for $i = 2, \dots, n$ to be all zero, is if all coordinates $x_2, \dots, x_n$ are zero; then considering the other two partial derivatives

$$\begin{cases} \frac{\partial}{\partial x_0}\overline{f} = d x_0^{d-1} + x_1^{d-1} = 0 \\ \frac{\partial}{\partial x_1}\overline{f} = (d-1) x_0 x_1^{d-2} = 0 \end{cases}$$

we get that also $x_0 = x_1 = 0$, that means, $\overline{Z}$ is smooth; furthermore, $\overline{Z}$ contains (at least) an $n-3$ dimensional family of lines, and whenever $d > n$ this is bigger than the expected dimension $2n - 3 - d$; this explains why the condition $d \leq n$ is necessary in the de Jong/Debarre conjecture.

□

Exercise 1.143. ?? Take $n = 2m + 1$ odd, and let $\Lambda \subset \mathbb{P}^n$ be an m-plane. Show that there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ of any given degree d containing Λ, and deduce once more that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Solution to Exercise ??: Consider coordinates $x_0, \dots, x_{2m+1}$ such that $\Lambda = (x_0, \dots, x_m)$ ■

Then, consider the polynomial

$$f(x) = \sum_{i=0}^{m} (x_i^d + x_i x_{i+m+1}^{d-1}.$$

This gives rise to a smooth hypersurface X (imposing all partial derivatives to zero, gives $m+1$ systems as the one in the previous exercise). So, $F_1(X)$ contains a $\mathbb{G}(1, m)$, that has dimension $2(m-2) = n-5$, that for $d > n+2$ is bigger than the expected dimension. □

Note that the construction of Exercise **??** cannot be modified to provide counterexamples to the de Jong/Debarre conjecture, since by Corollary **??** there do not exist smooth hypersurfaces $X \subset \mathbb{P}^n$ containing linear spaces of dimension strictly greater than $(n-1)/2$. The following exercise shows that the construction of Exercise **??** is similarly extremal, but is harder: it requires use of the second fundamental form of a hypersurface (see **?**]).

Exercise 1.144. **??** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that X can have at most finitely many hyperplane sections that are cones.

Solution to Exercise ??: The second fundamental form (one of its forms) on an hypersurface X is a map of vector bundles on X

$$g : \mathcal{N}^*_{X/\mathbb{P}^n} \to \mathrm{Sym}^2 TX^*$$

such that for every point p of X, the image in the fiber is the equation of the tangent cone at p of the hyperplane section $X \cap \mathbb{P}T_pX$; in particular, it is zero if p in $X \cap \mathbb{P}T_pX$ is a singular point of multiplicity ¿2. This map is obtained from the differential of the Gauss map to the dual hypersurface

$$G : X \to \mathbb{P}^{n*}$$

so in particular if g is globally zero, then G is globally constant.

Suppose now we have a curve C of points of X such that all hyperplane sections of points of C are cones; in particular, this means that g is zero at all these points, and hence that G is constant along the curve; this means that all points of C have the same tangent hyperplane H, and hence that $H \cap X$ is singular all along X. It is now an easy exercise to show that if an hyperplane section has positive dimensional singular locus, then the hypersurface itself cannot be smooth (all partial derivatives inside H vanish on the curve C, and the partial derivative that is normal to H imposes one condition, so that on finitely many points of C all partial derivatives vanish). □

To see some of the kinds of odd behavior the variety of lines on a smooth hypersurface can exhibit, short of having the wrong dimension, the following series of exercises will look at the Fermat quartic $X \subset \mathbb{P}^4$, that is, the zero locus

$$X = V(Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 + Z_4^4).$$

The conclusion is that $F_1(X)$ has 40 irreducible components, each of which is everywhere non-reduced! We start with a useful more general fact:

Exercise 1.145. **??** Let $S = \overline{p\,C} \subset \mathbb{P}^3$ be the cone with vertex p over a plane curve C of degree $d \geq 2$, and $L \subset S$ any line. Show that the tangent space $T_L F_1(S)$ has dimension at least two, and hence that $F_1(S)$ is everywhere nonreduced.

Solution to Exercise ??: Let us pick coordinates $x_0, \ldots, x_3$ for which $p = (1, 0, 0, 0)$, $L = (x_2, x_3)$ and the equation of C is $x_1^{d-1}x_3 + k$ where $k \in (x_2, x_3)^2$ (this will also be the equation for S in $\mathbb{P}^3$). Let us write local equations for $F_1(S)$ around L using coordinates a, b, c, d; being interested in Zariski tangent spaces, we will only care about linear terms in a, b, c, d; but the monomial $x_1^{d-1}x_3$ (the only one giving linear terms in a, b, c, d gives only the two equations $b = 0, d = 0$, so that the Zariski tangent space is 2 dimensional. But we also know that $F_1(S)$ is a 1-dimensional scheme (otherwise S would be a plane) so that $F_1(S)$ is everywhere nonreduced. □

Exercise 1.146. **??** Show that X has 40 conical hyperplane sections Y_i, each a cone over a quartic Fermat curve in $\mathbb{P}^2$.

Solution to Exercise ??: For every two variables Z_i and Z_j and every fourth root ζ of -1, we have that the hyperplane section $Z_i + \zeta Z_j = 0$ is a cone over a quartic (smooth) plane curve; there are 10 choices for the couples of variables, and 4 for the root of -1, that gives 40 choices total. □

Exercise 1.147. **??** Show that the reduced locus $F_1(Y_i)_{\text{red}}$ has class $4\sigma_{3,2}$.

Solution to Exercise ??: Y_i is a cone over a plane quartic curve, so that $F_1(Y_i)$ is one dimensional (and everywhere nonreduced, from Exercise **??**); to find the class $[F_1(Y_i)_{\text{red}}]$ we need to intersect with σ_1, that means, lines meeting a $\mathbb{P}^2$; but $Y_i \cap \mathbb{P}^2$ is generically 4 points, so the lines through these points are those meeting $\mathbb{P}^2$; we get then $[F_1(Y_i)_{\text{red}}] = 4\sigma_{3,2}$. □

Exercise 1.148. **??** Using your answer to Exercise **??**, conclude that

$$F_1(X) = \bigcup_{i=1}^{40} F_1(Y_i);$$

in other words, $F_1(X)$ is the union of 40 double curves.

Solution to Exercise ??: From Theorem **??**, we know that $F_1(X)$ is 1-dimensional, so it consists of the 40 nonreduced curves $F_1(Y_i)$ and possibly some other ones. Now, everyone of this curves has class $k\sigma_{3,2}$ where $k \geq 8$, and $k = 8$ only if these are just double curves. So, on one hand we get that the coefficient of $\sigma_{3,2}$ in $[F_1(X)]$ is bigger than or equal to $8 \cdot 40 = 320$. But $F_1(X)$ has the proper dimension, so we can calculate its class using Chern classes: using the splitting principle, we get $[F_1(X)] = 320\sigma_{3,2}$, that proves that $F_1(X)$ is only the union of the 40 curves $F_1(Y_i)$, and that they are all double. □

Exercise 1.149. **??** Show that

(a) There exist smooth quintic hypersurfaces $X \subset \mathbb{P}^5$ containing a 2-plane $\mathbb{P}^2 \subset \mathbb{P}^5$; and
(b) For such a hypersurface X, the family of conic curves on X has dimension strictly greater than the number $\lambda(5, 5, 2)$ of Conjecture **??**.

Solution to Exercise ??: For the first point, we can use Exercise**??**. For the second, a plane contains a 5 dimensional family of conics, and in this case $\lambda(5, 5, 2) = 3$, so the dimension is bigger; this tells in particular that when $d = n$ we cannot extend the de Jong/Debarre conjecture to the case of higher degree rational curves. □

1.7 Chapter 7

Exercise 1.150. **??** Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\{C_t \subset S\}_{t \in \mathbb{P}^1}$ be a general pencil of curves of type (a, b) on S, where $a, b > 0$. What is the expected number of curves C_t that are singular? (Make sure your answer agrees with (**??**) in case $(a, b) = (1, 1)$!)

Solution to Exercise ??: If all conditions of Proposition **??** are met, then the answer is the degree of $c_2(\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)))$, that is,

$$\begin{aligned} deg(3\lambda^2 + 2\lambda c_1 + c_2) &= deg(3(a\alpha + b\beta)^2 + 2(a\alpha + b\beta)(-2\alpha - 2\beta) + 4\alpha\beta) \\ &= 6ab - 4a - 4b + 4 \end{aligned}$$

where we used that

$$c(\Omega_{\mathbb{P}^1 \times \mathbb{P}^1}) = (1 + 2\alpha)(1 + 2\beta).$$

When $(a, b) = (1, 1)$, we get a general pencil of hyperplane sections of a quadric surface, so we expect the degree of the dual surface (so, 2) singular elements, that corresponds to the formula that we found. □

Exercise 1.151. **??** Prove that the number found in the previous exercise is the actual number of singular elements; that is, prove the three hypotheses of Proposition **??** in the case of $S = \mathbb{P}^1 \times \mathbb{P}^1$ and the line bundle $\mathcal{O}(a, b)$.

Solution to Exercise ??: If we have $(a, b) = (1, 1)$, then the three hypotheses hold: in fact, the only possible singular element of such linear series is the union of two lines, so with one single singularity, that is a simple double point. About point (c), this is obviously true for all (a, b). Consider now, in the linear series of (a, b), the curve C obtained by the union of a $(1, 1)$ curve that is singular at a point p (hence having a simple node), and any $(a-1, b-1)$ curve not passing through p. Consider now the linear system

$$V = \{f \in H^0(\mathcal{O}(a, b)) \mid f \text{ singular at } p\}$$

by Bertini, the general element is smooth away from the base locus of V, that is, away from p; but the element C of this linear series has a simple node at p, so the general element will too. We conclude that a general singular (a, b) curve has only one singularity that is a simple node. This same argument applies to prove the implication $e = 1$ to $e > 1$ in Section **??**. □

Exercise 1.152. **??** Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, and $L \subset S$ a line. Let $\{C_t\}_{t \in \mathbb{P}^1}$ be the pencil of conics on S cut out by the pencil of planes $\{H_t \subset \mathbb{P}^3\}$ containing L. How many of the conics C_t are singular? Use this to answer the question: how many other lines on S meet L?

Solution to Exercise ??: Again, we can invoke Proposition **??**, that gives us that the number is

$$deg(3\lambda^2 + 2\lambda c_1 + c_2).$$

Now, the first Chern class of $\mathcal{L}$ is $\lambda = \zeta - [L]$, and remember that $\deg(\zeta \cdot [L]) = 1$ and $\deg([L]^2) = -1$; moreover,

$$c(\Omega_S) = 1 - \zeta + 3\zeta^2$$

so that we get

$$\begin{aligned} deg(3\lambda^2 + 2\lambda c_1 + c_2) &= deg(3(\zeta - [L])^2 + 2(\zeta - [L])(-\zeta) + 3\zeta^2) \\ &= 3 \cdot 0 + 2(-2) + 9 = 5. \end{aligned}$$

To check that we are in the hypothesis of Proposition **??**, this time it is very easy because the element of the pencil is a conic curve, that can only degenerate as two incident lines (not as a double line, because a smooth cubic surface has not a nonreduced hyperplane section). This result has also consequences on the combinatorics of the 27 lines in S, because singular conics correspond to couple of lines that form a "triangular" hyperplane section of S together with L; using this, we can see that L meets 10 of the other lines on S. □

Exercise 1.153. **??** Let $p \in \mathbb{P}^2$ be a point, and let $\{C_t \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be a general pencil of plane curves singular at p—in other words, let F and G be two general polynomials vanishing to order 2 at p, and take $C_t = V(t_0 F + t_1 G)$. How many of the curves C_t will be singular somewhere else as well?

Solution to Exercise ??: In this problem we cannot use Proposition **??** on $\mathbb{P}^2$, because condition (c) is clearly not met. To avoid this issue, we can blow up $\mathbb{P}^2$ at the point p, and on the blowup X we can consider the proper transforms of the curves in the original pencil, and get a pencil of sections of $\mathcal{O}_X(d\zeta - 2E)$; now, condition (c) is met, and (a) and (b) descend from the $\mathbb{P}^2$ case. So, we can use the formula again: from

$$c(\Omega_X) = 1 - 3\zeta + E + 4\zeta^2$$

we get

$$deg(3(d\zeta - 2E)^2 + 2(d\zeta - 2E)(-3\zeta + E) + 4\zeta^2) = 3d^2 - 6d - 4$$

that is the number of curves having another singularity away from p. □

Exercise 1.154. **??** Let $S = X_1 \cap X_2 \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees e and f. If $\{H_t \subset \mathbb{P}^4\}_{t \in \mathbb{P}^1}$ is a general pencil of hyperplanes in $\mathbb{P}^4$, find the expected number of singular hyperplane sections $S \cap H_t$. (Equivalently: if $\Lambda \cong \mathbb{P}^2 \subset \mathbb{P}^4$ is a general 2-plane, how many tangent planes to S will intersect Λ in a line?)

Solution to Exercise ??: We invoke again Proposition **??**; remember that we have

$$c(\Omega_X) = \frac{(1-\zeta)^5}{(1-d\zeta)(1-e\zeta)} = 1 + (d+e-5)\zeta + (d^2+e^2+2de-5d-5e+10)\zeta^2$$

and we want to find

$$\begin{aligned} c_2(\mathcal{P}^1(1)) &= deg(3\zeta^2 + 2(d+e-5)\zeta^2 + (d^2+e^2+2de-5d-5e+10)\zeta^2) \\ &= de(d^2+e^2+2de-3d-3e+3). \end{aligned}$$

□

Exercise 1.155. **??** Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. Using formula (**??**), find the expected number of singular hyperplane sections of X in a pencil. Again, compare your answer to the result of Section **??**.

Solution to Exercise ??: To apply formula (**??**), we need

$$\begin{aligned} c(\Omega_X) &= \frac{(1-\zeta)^5}{(1-d\zeta)} = \\ &= 1 + (d-5)\zeta + (d^2-5d+10)\zeta^2 + (d^3-5d^2+10d-10)\zeta^3 \end{aligned}$$

and then we have

$$\begin{aligned} deg(c_3(\mathcal{P}^1(1))) &= deg(4\zeta^4 + 3(d-5)\zeta^4 + 2(d^2-5d+10)\zeta^4 + \\ &+ (d^3-5d^2+10d-10)\zeta^4) = \\ &= d(d^3-3d^2+3d-1) = d(d-1)^2 \end{aligned}$$

that is in fact the degree of the dual hypersurfaces as we found it in Section **??**. To check that the hypotheses of Proposition **??**, it is possible to follow the same was as for surfaces in $\mathbb{P}^3$ in Section **??** □

Exercise 1.156. **??** Let $X \cong \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ be the Segre threefold. Using formula (**??**), find the number of singular hyperplane sections of X in a pencil.

Solution to Exercise ??: Using Pformula (**??**), and the fact that

$$c(\Omega_X) = (1-2\alpha)(1-3\beta+3\beta^2) = 1-2\alpha-3\beta+6\alpha\beta+3\beta^2-6\alpha\beta^2,$$

we get

$$\begin{aligned} c_3(\mathcal{P}^1(1,1)) &= 4(\alpha+\beta)^3 + 3(\alpha+\beta)^2(-2\alpha-3\beta) + 2(\alpha+\beta)(6\alpha\beta+3\beta^2) - 6\alpha\beta^2 \\ &= 0. \end{aligned}$$

Note that condition (a) of Proposition **??** is not satisfied; in fact, the singular elements are (1, 1) are divisors splitting in a $\mathbb{P}^2$ fiber and in a $\mathbb{P}^1 \times \mathbb{P}^1$, meeting (hence being singular) along a line. Still, though, from the discussion after Proposition **??**, this implies that a general pencil has no singular elements. A diffenrent way to get this, is to prove that the dual variety of this Segre variety is not an hypersurface (it is in fact isomorphic to another Segre variety of the same kind), so a general pencil of hyperplanes does not contain any tangent one. A simple way to get this is just counting dimensions: the set of all possibilities for singular hyperplane sections, as union of a $\mathbb{P}^2$ and a $\mathbb{P}^1 \times \mathbb{P}^1$ is 3 dimensional, hence codimension 2 in $\mathbb{P}^{5*}$. See also Exercise **??** for the more general case of duals of scrolls. □

Exercise 1.157. **??** Let $S = X_1 \cap X_2 \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees e and f. What is the expected number of hyperplane sections of S having a triple point? (Check this in case $e = f = 2$!)

Solution to Exercise ??: For this exercise, we need to consider $\mathcal{P}^2(\mathcal{O}_S(1))$; we want the degeneracy locus among 5 sections, that will be given by c_2 of it. To find the Chern class of this bundle, let us first find the Chern classes of the vector bundle $\mathcal{E} = \mathcal{O} \oplus \Omega_S \oplus \mathrm{Sym}^2\,\Omega_S$; we know that the Chern classes of $\mathcal{P}^2(\mathcal{O}_S(1))$ are the same as those of $\mathcal{E} \otimes \mathcal{O}_S(1)$ (even though they are not isomorphic!) so in the end we will use Proposition **??** for Chern classes of tensor products by a line bundle. Using Examples **??**

and **??**, we get

$$c(\Omega_S) = \frac{1 - 5\zeta + 10\zeta^2}{(1 - e\zeta)(1 - f\zeta)} = 1 + (e + f - 5)\zeta + (e^2 + f^2 + ef - 5e - 5f + 10)\zeta^2$$

$$c(\mathrm{Sym}^2 \Omega_S) = 1 + 3(e + f - 5)\zeta + 2(3e^2 + 3f^3 + 4ef - 20e - 20f + 45)\zeta^2$$

$$c(\mathcal{E}) = 1 + 4(e + f - 5) + (10e^2 + 10f^2 + 15ef - 75e - 75f + 175)\zeta^2$$

and hence

$$\begin{aligned} c_2(\mathcal{P}^2(\mathcal{O}_S(1))) &= c_2(\mathcal{E} \otimes \mathcal{O}_S(1)) = c_2(E) + 5c_1(E)\zeta + 15\zeta^2 \\ &= (10e^2 + 10f^2 + 15ef - 55e - 55f + 90)\zeta^2 \end{aligned}$$

so that the degree is $de(10e^2 + 10f^2 + 15ef - 55e - 55f + 90)$; this is only the expected numbers: to prove that this is indeed the right number, one should try to extend Proposition **??** to this case. When $e = f = 2$, we get 40; this is the case of a Del Pezzo surface of degree 4; that means, the blow up of $\mathbb{P}^2$ at 5 points $p_1, \ldots, p_5$, embedded in $\mathbb{P}^4$ through the linear system of (proper transforms of) cubics through the 5 points. The elements with a triple point in this linear system are:

- the line through p_1, p_2, the line through p_3, p_4, and the line through the fifth point p_5 and the point q of intersection of the previous two lines (after permutations, there is 15 such curves);
- the conic through the 5 points, and the line tangent to this conic at one of the points p_1; in the blowup, these two curves and the exceptional divisor have a triple point (there is 5 of them);
- the line through p_1, p_2, and the conic through p_2, p_3, p_4, p_5 tangent to the line at p_2 (there is 20 of them).

In total, then, there is 40 of them. □

Exercise 1.158. **??** Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d whose general hyperplane section is a curve of genus g; let e and f be the degrees of the classes $c_1(\mathcal{T}_S)^2$ and $c_2(\mathcal{T}_S) \in A^2(S)$. Find the class of the cycle $T_1(S) \subset \mathbb{G}(1,n)$ of lines tangent to S in terms of d, e, f and g; from exercise **??**, we need only the intersection number $[T_1(S)] \cdot \sigma_3$.

Hint: consider instead the variety of tangent planes $T_2(S) \subset \mathbb{G}(2,n)$, and find the intersection with σ_2 as the intersection with $(\sigma_1)^2$ minus the intersection with $\sigma_{1,1}$.

Solution to Exercise ??: The intersection number $[T_1(S)] \cdot \sigma_3$ is the number of tangent lines that is tangent to S, that meet a general $\mathbb{P}^{n-4}$; this is the same as asking how many tangent planes to S meet the $\mathbb{P}^{n-4}$, that means, the intersection number $[T_2(S)] \cdot \sigma_2$ in $\mathbb{G}(2,n)$; remember that we have a map $S \xrightarrow{f} T_2(S)$, that we will suppose is birational.

As suggested in the hint, let us find it as the difference between intersections with σ_1^2 and $\sigma_{1,1}$. By push pull formula, the degree of the intersection $[T_2(S)]\cdot\sigma_1^2 \in A^{2n-2}(\mathbb{G}(1,n))$ is the same as the self intersection of the class $f^*\sigma_1$ in $A^2(S)$. But now, $[T_2(S)]\cdot\sigma_1$ are all tangent planes meeting a general $\mathbb{P}^{n-3}$; projecting away from that plane onto $\mathbb{P}^2$, these tangent planes correspond to points of S that are branch points for the restriction of the projection $\pi : S \to \mathbb{P}^2$; but this class is equal to $K_S - \pi^* K_{\mathbb{P}^2} = K_S + 3H$. Remember now that the degree of ζ^2 is d, that the degree of K_S^2 is e, and that by adjunction $\zeta(K_S + \zeta) = 2g - 2$; collecting everything, we get that the degree of $f^*\sigma_1^2$ is $3d + e + 12g - 12$, hence this is the degree of $[T_2(S)]\cdot\sigma_1^2$; to find the class $[T_2(S)]\cdot\sigma_{1,1}$, we need to find how many tangent planes meet a $\mathbb{P}^{n-2}$ in a line; that means, how many tangent planes are contained in one of the hyperplanes in the pencil of those containing the $\mathbb{P}^{n-2}$; that means, how many hyperplanes in that pencil cut a singular section on S; this is then a problem of singular elements in a linear series, and we need to find $c_2(\mathcal{P}^1(\mathcal{O}_S(1)))$; using Proposition **??**, we get that this number is $d + f + 4g - 4$. Collecting everything, we get that the degree of $[T_2(S)]\cdot\sigma_2$ (and hence of $[T_1(S)]\cdot\sigma_3$) is

$$2d + e - f + 8g - 8$$

and remembering also what we found in Exercise **??** (and using the indetermined coefficients method) we get

$$[T_1(S)] = (2d + e - f + 8g - 8)\sigma_{n-1,n-4} + (2d + 2g - 2)\sigma_{n-2,n-3}.$$

We will see alternative ways to solve these problems in Exercises **??** and **??**. □

Exercise 1.159. **??** Let $S \subset \mathbb{P}^3$ be a general surface of degree d, and $\mathcal{B}$ a general net of plane sections of S (that is, intersections of X with planes containing a general point $p \in \mathbb{P}^3$). What are the degree and genus of the curve $\Gamma \subset S$ traced out by singular points of this net? What are the degree and genus of the discriminant curve? Use this to describe the geometry of the finite map $\pi_p : S \to \mathbb{P}^2$ given by projection from p.

Solution to Exercise ??: Following the discussion in **??**, let us consider the curve $\Sigma_{\mathcal{B}} \subset \mathcal{B}\times S$ (that is smooth by Bertini). Let us find its class; calling $E = \pi_2^*(\mathcal{O}_S \oplus \Omega_S)$, the class we need is $c_3(\mathcal{O}_{\mathcal{B}\times S}(1,1) \otimes E)$ (remember that this vector bundle is *not* isomorphic to the bundle of principal parts that we are using, they only share Chern classes); using all the required formulas, we get in the end

$$[\Sigma_{\mathcal{B}}] = (d-1)\zeta_{\mathcal{B}}^2\zeta_S + (d-1)^2\zeta_{\mathcal{B}}\zeta_S^2,$$

where we used that $\zeta_{\mathcal{B}}^3 = \zeta_S^3 = 0$ Note that this formula reflects the fact that if $d = 1$, that is if S is a plane, there is no singular plane section so the class is zero. To find the genus of $\Sigma_{\mathcal{B}}$, we can use adjunction as in the proof of Proposition **??**; we then get

$$\begin{aligned}2g-2 &= deg([\Sigma_{\mathcal{B}}]\cdot(-c_1(T_{\mathcal{B}}\times S)+c_1(\mathcal{O}_{\mathcal{B}\times S}(1,1)\otimes E)) = \\ &= deg([\Sigma_{\mathcal{B}}]\cdot(-(3\zeta_{\mathcal{B}}+(4-d)\zeta_S)+(3\zeta_{\mathcal{B}}+(d-1)\zeta_S)) = \\ &= deg([\Sigma_{\mathcal{B}}]\cdot(2d-5)\zeta_S) = (d-1)(2d-5)d\end{aligned}$$

where we used that the degree of $\zeta_{\mathcal{B}}^2\zeta_S^2$ is d; we then find $g = \frac{(d-2)(2d^2-3d-1)}{2}$. To find the degree of the image $\Gamma = \pi_2(\Sigma_{\mathcal{B}})$, by push pull we just need to find the degree of $[\Sigma_{\mathcal{B}}]\cdot\zeta_S$, that is $d(d-1)$ (note that from Exercise **??** we have that the $\pi_2 : \Sigma_{\mathcal{B}} \to \Gamma$ is birational, so the two curves have the same geometric genus). To find the arithmetic genus of $\pi_2(\Sigma_{\mathcal{B}})$, we can use adjunction, remembering that the class of Γ in $A^1(S)$ is $c_1(\mathcal{P}^1(1)) = (d-1)\zeta_S$; we then get

$$\begin{aligned}2g_\Gamma - 2 &= deg([\Gamma]\cdot([\Gamma]+c_1(K_S)) = \\ &= deg(d-1)\zeta_S\cdot(2d-5)\zeta_S = (d-1)(2d-5)d\end{aligned}$$

that gives $g_\Gamma = g$, so for Γ arithmetic and geometric genus coincide, so we can conclude that Γ is smooth. About the discriminant curve $\mathcal{D} = \pi_1(\Sigma_{\mathcal{B}})$, its degree, by the push pull formula again, is given by $d(d-1)^2$, hence its arithmetic genus is $\binom{d(d-1)^2-1}{2}$, that is very different from the geometric one, unless $d = 2$ where they are the same; we get again the statement that the discriminant curve is smooth if and only if $d = 2$. The geometric genus of $\mathcal{D}$ is still the same of $\Sigma_{\mathcal{B}}$, because the general singular hyperplane section is singular only at one point (see Section **??**).

All of this is related to the geometry of the projection π of S away from the basepoint p of the net $\mathcal{B}$ onto $\mathbb{P}^2$ (that can be identified with the dual of $\mathcal{B}$); points of Γ will be ramification points of π on S, and the branch locus on $\mathbb{P}^2$ will be the dual curve $\mathcal{D}*$. □

Exercise 1.160. **??** Verify that for a general curve $C \subset \mathbb{P}^2$ of degree d the number $3d(d-2)$ is the actual number of flexes of C; that is, all inflection points of C will have weight 1.

Solution to Exercise ??: Let us consider the "universal hyperflex point-line" correspondence

$$\Phi = \{(p, L, C) \in \mathbb{P}^2 \times \mathbb{P}^{2*} \times \mathbb{P}^N \mid m_p(C, L) \geq 4\}.$$

If the image of Φ under the projection onto $\mathbb{P}^N$ is not dominant, then the general plane curve will not have any hyperflex (that means, no flexes of weight 2 or more). But considering the projection onto the 3-dimensional correspondence point-line in $\mathbb{P}^2\times\mathbb{P}^{2*}$, fibers are linear subspaces of $\mathbb{P}^N$ of codimension 4, so Φ has dimension $N-1$ that proves the claim. □

Exercise 1.161. **??** Let $\{C_t \subset \mathbb{P}^2\}_{t\in\mathbb{P}^1}$ be a general pencil of plane curves of degree $d \geq 3$; suppose C_0 is a singular element of C (so that in particular by Proposition **??**,

C_0 will have just one node as singularity). By our formula, C_0 will have 6 fewer flexes than the general member C_t of the pencil. Where do the other 6 flexes go? If we consider the incidence correspondence

$$\Phi = \{(t, p) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid C_t \text{ is smooth and } p \text{ is a flex of } C_t\},$$

what is the geometry of the closure of Φ near $t = 0$? Bonus question: describe the geometry of

$$\tilde{\Phi} = \{(t, p, L) \in \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^{2*} \mid C_t \text{ smooth, } p \text{ a flex of } C_t \text{ and } L = \mathbb{T}_p C_t\}$$

near $t = 0$.

Solution to Exercise ??: Let us first deal with $\tilde{\Phi}$; the statements about Φ will come as consequence. Note that for $t \neq 0$, in every element (t, p, L) we have $m_p(C_t, L) \geq 3$; this is a closed condition, so it will hold also on the limit; on C_0, then, this condition is satisfied for the "honest" flexes, but also for the node n with the two tangent lines M_1 and M_2; going in local coordinates it is possible to prove that every honest flex on C_0 is limit of at most one flex of neighbor fibers (otherwise it would become an hyperflex, and a general nodal curve does not have any); so, 6 flexes of fibers C_t where $t \neq 0$ degenerate to either $(0, n, M_1)$ or $(0, n, M_2)$; this tells us immediately that the curve Φ has an order 6 ramification point over 0. With an argument of monodromy (see **?**]) it is possible to prove that $\tilde{\Phi}$ has two order 3 ramification points at $(0, n, M_1)$ or $(0, n, M_2)$, so that the 6 remaining flexes are distributed as 3 and 3 on the two branches at the node. □

Exercise 1.162. **??** Find the points on $\mathbb{P}^1$, if any, that are ramification points for the maps $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$(s, t) \mapsto (s^3, s^2t, st^2, t^3) \in \mathbb{P}^3$$

and

$$(s, t) \mapsto (s^4, s^3t, st^3, t^4) \in \mathbb{P}^3.$$

Solution to Exercise ??: For the first, Plücker formula tells us that there are no ramification points. For the second, the formula gives 4 as sum of weight of inflexionary points; note that an inflexionary point means a plane with order of contact at least 4, so we could not really expect any inflexionary point from a degree 3 curve. To find these points, let us trivialize the map $(\mathcal{O}^1_{\mathbb{P}})^{\oplus 4} \to \mathcal{P}^3(4)$ setting $s = 1$; as in the proof of Theorem **??**, inflexionary point are given by the determinant of

$$\begin{pmatrix} 1 & t & t^3 & t^4 \\ 0 & 1 & 3t^2 & 4t^3 \\ 0 & 0 & 6t & 12t^2 \\ 0 & 0 & 6 & 24t \end{pmatrix}$$

that is $72t^2$, so that we have an inflexionary point of order 2 at $[1, 0]$; changing chart and setting $t = 1$, we get also an inflexionary point of weight 2 at $[0, 1]$. □

Exercise 1.163. ?? Show that the only smooth, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ with no inflection points is the rational normal curve.

Solution to Exercise ??: From Plücker formula, we need the number $(r + 1)(d + rg - r) = 0$; this happens if and only if $g = 0$ and $r = d$, that means a rational normal curve (notice that a degenerate curve does not come from the embedding of a linear series so it is not allowed here). □

Exercise 1.164. ?? Observe that in case $g = 1$ and $d = r + 1$—that is, the curve is an elliptic normal curve E—the Plücker formula yields the number $(r + 1)^2$ of inflection points. Show that these are exactly the translates of any one by the points of order $r + 1$ on E, each having weight 1.

Solution to Exercise ??: A point p is inflexionary if and only if there is a divisor in the linear series containing a multiple of p bigger than or equal than $r + 1$; in this case the degree is $r + 1$, so p is inflexionary if and only if $(r + 1)p$ is in the linear series. If p is inflexionary, all its traslates by an $r + 1$-torsion point are still inflexionary (because the embedding is normal); so there are $(r + 1)^2$ different inflexionary points, so they are all of weight one. □

Exercise 1.165. ?? Let C be a smooth curve of genus $g \geq 2$. A point $p \in C$ is called a *Weierstrass point* if there exists a nonconstant rational function on C with a pole of order g or less at p and regular on $C \setminus \{p\}$.

(a) Show that the Weierstrass points of C are exactly the inflection points of the canonical map $\varphi : C \to \mathbb{P}^{g-1}$; and
(b) Use this to count the number of Weierstrass points on C.

Solution to Exercise ??: Let p be an inflexionary point of the canonical map; this means that there is an hyperplane section (i.e. a differential form) of the kind $gp + D$ where D is a positive divisor of degree $g - 2$; this means, $h^0(K - gp) \geq 0$, so using the Riemann-Roch formula we get

$$h^0(gp) = 1 + h^0(K - gp) \geq 1$$

that is exactly the condition for the existence of such a rational function. Using Pl"ucker formula, we get that the sum of the weights of Weiestrass points of a curve is $g^3 - g$. □

Exercise 1.166. ?? Let $\mathbb{P}^N$ be the space of all plane curves of degree $d \geq 4$, and let $H \subset \mathbb{P}^N$ be the closure of the locus of smooth curves with a hyperflex. Show that H is a hypersurface. (We'll be able to calculate the degree of this hypersurface once we have developed the techniques of Chapter **??**.)

Solution to Exercise ??: Looking back at Exercise **??**, we only need to prove that the projection from the universal hyperflex onto $\mathbb{P}^N$ is generically finite. From the irreducibility of Φ, we only need to show one curve exhibiting a single isolated hyperflex; consider now the Fermat curve, the curve C given by the equation $x_0^4 + x_1^4 + x_2^4 = 0$. Intersecting with the Hessian curve, it has 12 isolated flexes (and it is easy to show they have all weight 2). When the degree is higher, it is sufficient to consider the union of C and any other curve not passing through flexes of C; this is still going to have isolated hyperflexes. □

Exercise 1.167. To prove that a general complete intersection $C \subset \mathbb{P}^3$ does not have weight two inflectionary points, we need to prove that it does not have flex lines (lines with multiplicity 3 intersection with the curve) or planes with a point of contact of order 5. Prove the first statement, that a general complete intersection of two surfaces S_1 and S_2 of degrees $d_1 \geq d_2 > 1$ does not have a flex line.

Solution to Exercise 1.167: Let us consider the incidence correspondence Φ inside

$$\mathbb{P}^3 \times \mathbb{G}(1,3) \times \mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$$

such that elements are (p, L, F_1, F_2) such that the surfaces given by the polynomials F_1 and F_2 intersect transversely in a (complete intersection) curve C, and L is a flex curve of C at the point p. Note that this subscheme is not closed, and that the same curve appears plenty of time; in particular, when F_1 changes by a multiple of F_2, so this projects down onto the space H_{d_1,d_2} parametrizing complete intersections of multidegree d_1, d_2 with fibers of dimension $\binom{d_1-d_2+3}{3}$. Given a couple (p, L), to get a curve C with a flex there we need to have both surfaces to vanish along L at p with multiplicity 3; so, it imposes three indipendent linear conditions on $\mathbb{P}^{N_1}$ and $\mathbb{P}^{N_2}$ (this is because both d_1 and d_2 are greater than 1, otherwise this wouldn't be true). So, the dimension of Φ is $N_1 + N_2 - 1$; now, Φ also projects onto H_{d_1,d_2} with fibers of dimension $\binom{d_1-d_2+3}{3}$ (because also in Φ we can change F_1 by a multiple of F_2), so it has image of codimension 1; this proves that the general complete intersection does not have a flex line. Note that the condition $d_2 > 1$ is necessary: otherwise, we would have a plane curve, that for $d_1 \geq 3$ has flexes (hence flex lines). The spaces H_{d_1,d_2} will be introduced and described in the next chapter. □

Exercise 1.168. **??** Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 2$, $p \in S$ a general point and $H = \mathbb{T}_p S \subset \mathbb{P}^3$ the tangent plane to S at p. Show that the intersection $H \cap S$ has an ordinary double point at p.

Solution to Exercise ??: We need to consider the incidence correspondence

$$\Phi = \overline{\{(p, H, S) \in \mathbb{P}^3 \times \mathbb{P}^{3*} \times \mathbb{P}^N \mid H = T_p X, X \cap T_p X \text{ cusp at } p\}}$$

that is, a universal point-plane correspondence for hyperplane sections that have not an ordinary double point. Projecting onto the 5-dimensional incidence in $\mathbb{P}^3 \times \mathbb{P}^{3*}$, fibers are irreducible degree two hypersurfaces in codimension 3 linear subspaces of $\mathbb{P}^N$, so that Φ is irreducible and of dimension $N+1$; when projecting onto the $N+2$-dimensional universal point correspondence in $\mathbb{P}^3 \times \mathbb{P}^N$, the map from Φ cannot be dominant. We then proved that the at the general point of the general surface of degree d, the plane section has an ordinary double point. □

Exercise 1.169. **??** Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\{C_t \subset S\}_{t \in \mathbb{P}^1}$ be a general pencil of curves of type (a, b) on S. Use the topological Hurwitz formula to say how many of the curves C_t are singular. (Compare this with your answer to Exercise **??**.)

Solution to Exercise ??: A pencil of (a, b) curves is going to have $2ab$ basepoints, so that the total space is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $2ab$ points, hence having Euler characteristic $\chi(X) = 4 + 2ab$. The general fiber is going to have genus $(a-1)(b-1)$, so that $\chi(C_\eta) = 2a + 2b - 2ab$. We then have

$$\delta = \chi(X) - \chi(\mathbb{P}^1)\chi(C_\eta) = 6ab - 4a - 4b + 4$$

that is the same that we found in Exercise **??**. □

Exercise 1.170. **??** Let $p \in \mathbb{P}^2$ be a point, and let $\{C_t \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be a general pencil of plane curves of degree d singular at p, as in Exercise **??**. Use the topological Hurwitz formula to count the number of curves in the pencil singular somewhere else.

Solution to Exercise ??: The base points of the pencil are the point p and $d^2 - 4$ other general points in $\mathbb{P}^2$; the total space will then be the blow up of $\mathbb{P}^2$ at $d^2 - 3$, so that $\chi(X) = d^2$. The arithmetic genus of the general curve is $\binom{d-1}{2} - 1$, so that $\chi(C_\eta) = -d^2 + 3d + 2$. We then have

$$\delta = \chi(X) - \chi(\mathbb{P}^1)\chi(C_\eta) = 3d^2 - 6d - 4$$

that is the same as in Exercise **??**. □

Exercise 1.171. **??** Let $\mathbb{P}^5$ be the space of conic plane curves, and $\mathcal{D} \subset \mathbb{P}^5$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double line. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?

Solution to Exercise ??: Consider a general pencil of conics containing C; the base locus will be two degree 2 schemes supported in one point, and so there will be only another singular element in the fiber, composed by the two lines spanned by this two schemes. The degree of the discriminant hypersurface is 3, so it follows that $\mathrm{mult}_C(\mathcal{D}) = 2$. To find the tangent cone, we need to find the pencils for which the above situation does not happen; this is exactly the case where the base locus is supported in one point p; in this case, we have families only including conics that are tangent to the line where

C is supported; this is going to be the tangent cone (that is in fact a cone over a singular quadric). □

Exercise 1.172. **??** Let $\mathbb{P}^{14}$ now be the space of quartic plane curves, and $\mathcal{D} \subset \mathbb{P}^{14}$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double conic. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?

Solution to Exercise ??: We will use a different way (that in fact gives a different proof of the previous exercise). Consider a general pencil $\{t_0 F + t_1 Q^2\}$ where Q^2 is a double conic and F is a general quartic; notice that the base locus, the intersection of F and Q^2 is eight double points. The total space will then be the blow up of $\mathbb{P}^2$ at 8 points; we cannot use still the topological Hurwitz formula, because the space obtained in this way is still singular at the 8 points on the fiber Q^2 (this happens whenever there is a nonreduced base locus, so also in the case of the previous exercise). So, let us blow up *again* at these 8 points; now, the fibers are going to be all curves in our pencil, besides Q^2 that now has become the union C_0 of a rational curve (the conic itself) with the 8 new exceptional curves; that means, with topological Euler characteristic 10. Now, we can use the topological Hurwitz formula; the fibers with a different Euler characteristic are δ curves with a node, and the central fiber $Q^2 \cup \bigcup_{i=1}^{8} E_i$. We then get

$$\delta = \chi(X) - \chi(C_t)\chi(\mathbb{P}^1) - (\chi(C_0) - \chi(C_t)) = 19 - (-4) \cdot 2 - 14 = 13$$

so only other 13 singular fibers; the degree of the discrimimant hypersurface is 27, so the double conic must be multiplicity 14. Notice that this is the smallest multiplicity that implies that every pencil containing two double conics is entirely contained in the discriminant hypersurface. The tangent cone will be composed by quartics that are tangent to the conics; it is easy to see that this is a degree 14 hypersurface, confirming again that the multiplicity at double conics is 14. □

1.8 Chapter 8

Exercise 1.173. **??** Let $D \subset \mathbb{P}^2$ be a smooth curve of degree d, and let $Z \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z.

Solution to Exercise ??: Following Lemma **??**, the class will be $[Z_D] = p\alpha + q\beta$; we will use instead a different basis for the Picard group, using the fact that the space of complete conics is the same as the blowup of $\mathbb{P}^5$ along the Veronese surface of double lines; we will use the hyperplane class α of $\mathbb{P}^5$ and the exceptional divisor ϵ (that is $2\alpha - \beta$ as we see in the end of Section **??**). We can then write $[Z_D] = u\alpha + v\beta$; intersecting with a general pencil of conics in $\mathbb{P}^5$ (that in Lemma **??** is called γ), we get a pencil of degree $2d$ divisors on D, that have, byRiemann-Hurwitz theorem, $4d + 2g - 2 = d^2 + d$ singular elements (that correspond to conics that are tangent to D): this will be the coefficient u. To find the coefficient of ϵ, we need to find the multiplicity of Z_D in $\mathbb{P}^5$ along S; considering a general pencil including a double line, we see that the linear system on D has one fiber that is d double points (the intersection with the double line), and d^2 other ramifications: by an argument as in **??**, this proves that this multiplicity is d; we then get

$$[Z_D] = d(d+1)\alpha - d\epsilon = d(d-1)\alpha + d\beta$$

going back to the other basis. □

Exercise 1.174. **??** Now let $D_1, \ldots, D_5 \subset \mathbb{P}^2$ be general curves of degrees $d_1, \ldots, d_5$. Show that the corresponding cycles $Z_{D_i} \subset X$ intersect transversely, and that the intersection is contained in the open set U of smooth conics.

Solution to Exercise ??: The fact that the intersection lies in the open set of smooth conics can be proved in the exact same way as if $D_1, \ldots, D_5$ were conics, as it is done in Section **??**. Given a point C in the cycle Z_D, let us find the tangent space $T_C Z_{D_i}$; a general pencil of conics containing C induces a degree $2d$ linear system on D with $d^2 + d - 1$ other singular elements (that are conics tangent to D); XXX □

Exercise 1.175. **??** Combining the preceding two exercises, find the number of smooth conics tangent to each of five general curves $D_i \subset \mathbb{P}^2$.

Solution to Exercise ??: After the two preceding exercises, we only need to find the intersection product of the classes $[Z_{D_i}] = d_i(d_i - 1)\alpha + d_i\beta$. Calling τ_r the rth symmetric function of the 5 numbers $d_1, \ldots, d_5$, that means,

$$\tau_i = \sum_{|I|=r} \prod_{i \in I} d_i$$

we have

$$deg \prod_i (d_i(d_i - 1)\alpha + d_i\beta) = deg(\prod_i d_i(\sum_I \prod_{i \in I} (d_i - 1)\alpha^{|I|}\beta^{5-|I|})) =$$

$$\tau_5(\tau_5 + \tau_4 + \tau_3 - 3\tau_2 + 3\tau_1)$$

that in fact agrees with what we already know about conics tangent to 5 lines (1) and conics tangent to 5 conics (3264). For instance, we get that the number of conics tangent to 5 general cubics is 168399. □

Exercise 1.176. **??** Let $D \subset \mathbb{P}^2$ be a curve of degree d with δ simple nodes, κ simple cusps (for a definition of cusps, see Section **??**) and smooth otherwise. Let $Z \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D at a smooth point of D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z_D.

Solution to Exercise ??: It is easy to see that conics in Z_D that we are adding when taking the closure are conics through the nodes of the cusps of D that are tangent to the branches (the single branch in case of the cusps). Let us find the class $[Z_D] = u\alpha + v\epsilon$. The coefficient u we need to find the intersection with a pencil of conics through 4 points; consider the normalization $\tilde{D}$ of D (that has genus $g = \binom{d-1}{2} - \delta - \kappa$) and the pencil of conics induces a linear system of degree 2d. From Riemann-Hurwitz formula, the number of singular elements of the series is $4d + 2g - 2 = d^2 + d - 2\delta - 2\kappa$; there will be a conic in the pencil through every node of D, but that will not give a singular element in $\tilde{D}$, because if the pencil is general it will not be tangent to any of the two branches, so on $\tilde{D}$ it lifts to two reduced points on each point in the inverse image of the node. There is a conic through every of the cusps of D, again that is not tangent to the branch; this is going to be a ramification point for the system on $\tilde{D}$ (it is easy to see in local coordinates that it is a simple ramification point); but, we do not want to count these conics, because they are not in Z_D; so, the number we need is

$$u = 4d + 2g - 2 - \kappa = d^2 + d - 2\delta - 3\kappa.$$

Note that this number is also the degree of the curve dual to D in $\mathbb{P}^{2*}$. To find the coefficient v, it is d exactly as in Exercise **??**. The class is then

$$[Z_D] = (d^2 + d - 2\delta - 3\kappa)\alpha + d\epsilon = (d^2 - d - 2\delta - 3\kappa)\alpha + d\beta.$$

□

Exercise 1.177. **??** Let $\{D_t\}$ be a family of plane curves of degree d, with D_t smooth for $t \neq 0$ and D_0 having a node at a point p. What is the limit of the cycles Z_{D_t} as $t \to 0$?

Solution to Exercise ??: For $t \neq 0$, cycles Z_{D_t} can be defined as the set of conics that have a double point of intersection with D_t; this will hold also in the limit, so the support of the limit will be contained in the set of conics that are either tangent to D_0 at a smooth point, or that pass through the node p, that means, $Z_{D_0} \cup A_p$; the multiplicities along each of the two component can be checked using the fact that the total class has to be the same as $[Z_{D_t}] = d(d-1)\alpha + d\beta$; so, using the previous exercise, the limit cycle has multiplicity one on Z_{D_0} and multiplicity two along A_p. □

Exercise 1.178. **??** Here's a very 19th century way of deriving the result of Exercise **??** above. Let $\{D_t\}$ be a pencil of plane curves of degree d, with D_t smooth for general t and D_0 consisting of the union of d general lines in the plane. Using the description of the limit of the cycles Z_{D_t} as $t \to 0$ in the preceding exercise, find the class of the cycle Z_{D_t} for t general.

Solution to Exercise ??: Let $L_1, \ldots, L_d$ be the lines composing D_0, and $p_{12}, p_{13}, \ldots, p_{d-1,d}$

its nodes. Using the previous exercise, the cycle Z_{D_t} degenerates to the union of cycles B_{L_i} with multiplicity one, and the cycles $A_{p_{ij}}$ with multiplicity 2. Summing up everything, we get

$$[Z_{D_t}] = 2\binom{d}{2}\alpha + d\beta$$

as we found in Exercise **??**. □

Exercise 1.179. **??** True or False: There are only finitely many PGL_4 orbits in the Kontsevich space $\overline{M}_0(\mathbb{P}^3, 3)$.

Solution to Exercise ??: False. Let us consider the most degenerate situation; as in the case of plane conics, we consider three-to-one maps of a rational curve onto a line in $\mathbb{P}^3$; such elements are completely determined by the image line $L \subset \mathbb{P}^3$ and the 4 branch points on L; PGL_4 acts on these configurations, but it cannot change the cross ratio of the four points. There is still a one dimensional parameter that distinguish orbits, so (unless we are dealing with a finite field) the statement is false. □

Exercise 1.180. **??** Let Γ_1 and Γ_2 be collections of $3d_1 - 1$ and $3d_2 - 1$ general points in $\mathbb{P}^2$, and $D_i \subset \mathbb{P}^2$ any of the finitely many rational curves of degree d_i passing through Γ_i. Show that D_1 and D_2 intersect transversely.

Solution to Exercise ??: Let us consider the incidence correspondence

$$\Phi \subset (\mathbb{P}^2)^{(3d_1-1)} \times (\mathbb{P}^2)^{(3d_2-1)} \times \overline{M}_0(\mathbb{P}^2, d_1) \times \overline{M}_0(\mathbb{P}^2, d_1) \times \mathbb{P}^2 \times \mathbb{P}^{2*}$$

where the elements

$$(\Gamma_1, \Gamma_2, C_1, C_2, p, L)$$

are in such a way C_i contains Γ_i, p is one of the intersection points of $C_1 \cap C_2$, and L is the tangent line of both C_1 and C_2 at p. Let us find the dimension of Φ, and see that it has dimension smaller than $(\mathbb{P}^2)^{(3d_1-1)} \times (\mathbb{P}^2)^{(3d_2-1)}$ (that is $6d_1 + 6d_2 - 4$ dimensional), so that for general Γ_1 and Γ_2 we do not have two tangent curves. Let us start considering the image Ψ of Φ onto $\overline{M}_0(\mathbb{P}^2, d_1) \times \overline{M}_0(\mathbb{P}^2, d_1) \times \mathbb{P}^2 \times \mathbb{P}^{2*}$, by projecting it again to the incidence correspondence $\Xi \in \mathbb{P}^2 \times \mathbb{P}^{2*}$. Every $(p, L) \in \Xi$ imposes two conditions on curves in $\overline{M}_0(\mathbb{P}^2, d_i)$, so fibers are $3d_i - 3$ dimensional, so Ψ has dimension $3d_1 + 3d_3 - 3$. Now, fibers of Φ over Ψ are just choices of $3d_i - 1$ points on the curves C_i, so dimension of fibers is $3d_1 + 3d_3 - 2$, and the dimension of Φ is $6d_1 + 6d_2 - 5$, that hence cannot dominate $(\mathbb{P}^2)^{(3d_1-1)} \times (\mathbb{P}^2)^{(3d_2-1)}$. □

Exercise 1.181. **??** Let $p_1, \ldots, p_7 \in \mathbb{P}^2$ be general points and $L \subset \mathbb{P}^2$ a general line. How many rational cubics pass through $p_1, \ldots, p_7$ and are tangent to L?

Solution to Exercise ??: Let us follow the discussion in Section **??**; at first, we need to better study the cycle in $\overline{M}_0(\mathbb{P}^2, 3)$ of cubics tangent to the line L; a couple of remarks: this cycle does not include cubics whose singular point is on the line L (because it cannot be smoothed out); then, a reducible element with a node p on L (and no further tangency condition) is in the cycle if and only if p is also a node on the source curve. Then, in the same way as in Section **??**, we consider the curve B where we substitute one of the condition of containing one of the points of Γ to the condition of being tangent to L; we still have that curves in B break in at most two components (that can be only one line and a conic in this case); then we can just follow the discussion; the only mayor difference is that when the curve splits in two components, we have to both consider the possibility of the conic being tangent to L, and the possibility of the node (the one that is still a node in the source curve) lying on L, and one should remember that in the former case the number of conics through 4 points and tangent to a line are 2. In the end, the number is 48, and in fact it can be verified also in another way; inside the space $\mathbb{P}^9$ of cubic curves in $\mathbb{P}^2$, we need to intersect the degree 12 hypersurface of singular curves, the degree 4 hypersurface of curves tangent to a line, and 7 hyperplanes; the first step is to prove that the intersection indeed happens on curves with at most one nodal singularity (so that the intersection is the same as in $\overline{M}$), than using the explicit description of tangent spaces (that follows from Proposition **??** and an argument as in Exercise **??**) it is possible to prove that they intersect transversely, so in 48 points. □

Exercise 1.182. ??

(a) Let $M = \overline{M}_0(\mathbb{P}^2, d)$ be the Kontsevich space of rational plane curves of degree d, and let $U \subset M$ be the open set of immersions $f : \mathbb{P}^1 \to \mathbb{P}^2$ that are birational onto their images. For $D \subset \mathbb{P}^2$ a smooth curve, let $Z_D^\circ \subset U$ be the locus of maps $f : \mathbb{P}^1 \to \mathbb{P}^2$ such that $f(\mathbb{P}^1)$ is tangent to D at a smooth point of $f(\mathbb{P}^1)$, and $Z_D \subset M$ its closure. Verify the statement above: that is, show that Z_D is contained in the locus of maps $f : C \to \mathbb{P}^r$ such that the preimage $f^{-1}(D)$ is nonreduced or positive-dimensional.

(b) Given this, show that for $D_1, \ldots, D_{3d-1}$ general curves the intersection $\cap Z_{D_i}$ is contained in U.

Solution to Exercise ??: It is easy to prove that the condition of $f^{-1}(D)$ being nonreduced or positive-dimensional is a closed condition; the closure of Z_D has then to be contained in it. XXXASK SOMEONE Suppose now a stable map $\{f : C \to \mathbb{P}^2\}$ is in Z_D but not in U; this means that it has a rational component $C_0 \subset C$ that is contracted to a point p in $\mathbb{P}^2$. In order for the map to be stable, we need C_0 to meet at least 3 other components C_1, C_2, C_3 of C, that map through f to curves of degree d_1, d_2, d_3 with $d_1 + d_2 + d_3 \leq d$. We know that $f(C_1), f(C_2), f(C_3)$ have to pass to p, and they need to be tangent to $3d - 3$ curves (we can avoid two if we impose p to be the intersection of two of the curves, not three because they are general). Note also that if two of the curves $f(C_i)$ intersect somewhere else than p along one of the curves D_i, the preimage will be two reduced points on the curves C_i unless one of the two curves in $\mathbb{P}^2$ is indeed tangent to the curve D_i. So, we have cumulatively $3d$ conditions on the three curves, but moduli spaces for curves of degrees d_1, d_2, d_3 have cumulative dimension $3d_1 + 3d_2 + 3d_3 - 3$ that is smaller than the number of conditions, $3d$, so such a curve cannot exist. XXXCHECK AGAIN □

1.9 Chapter 9

Exercise 1.183. **??** Choosing coordinates $x_0, x_1, \ldots, x_a$ on $\mathbb{P}^a$ to correspond to the monomials $s^a, s^{a-1}, \ldots, t^a$, show that the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} \\ x_1 & x_2 & \ldots & x_a \end{pmatrix}$$

vanish identically on the rational normal curve $S(a)$. By working in local coordinates, show that the ideal I generated by the minors defines the curve scheme theoretically. Find a set of monomials that form a basis for the ring $K[x_0, x_1, \ldots, x_a]/I$, and show that in degree d it has dimension $ad + 1$. By comparing this with the Hilbert function of $\mathbb{P}^1$, prove that I is the saturated ideal of the rational normal curve.

Solution to Exercise ??: Plugging in coordinates s, t in the matrix, it has rank 1, so all minors vanish on the curve. Let us prove that the 2×2 minors define the curve; it is easy to check that it defines it set theoretically; to say that the curve is the scheme cut out by these polynomials, we will prove that the vanishing locus has tangent space of dimension one at every point of the curve. At the point $[1, 0, \ldots, 0]$, the equations $x_0 x_{i+1} - x_1 x_i$ say that the tangent space to the vanishing locus is where the coordinates $x_2, \ldots, x_n$ vanish, that means, a line. At the point $[s^a, s^{a-1}t, \ldots, t^a]$ with $t \neq 0$, we slightly change the matrix into

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} \\ sx_1 - tx_0 & sx_2 - tx_1 & \ldots & sx_a - tx_{a-1} \end{pmatrix}$$

that has rank 1 if and only if the previous one has, so that the minors generate the same ideal; now, if we consider t times the minor given by the columns i and $i + 1$, we get

$$tx_{i-1}(sx_{i+1} - tx_i) - tx_i(sx_i - tx_{i-1}) = 0$$

$$x_i(t^2 x_{i-1} - 2stx_i + s^2 x_{i+1}) - (sx_{i+1} - tx_i)(sx_i - tx_{i-1}) = 0$$

that proves that the tangent space is the vanishing locus of the linear forms $t^2 x_{i-1} - 2stx_i + s^2 x_{i+1}$ as i varies from 1 to $a - 1$, that means a line. For the last part of the problem, a basis for the coordinate ring can be given by all monomials $x_i^e x_{i+1}^f$, that it is easy to see are all independent modulo I, and together with I generate the entire $K[x_0, x_1, \dots, x_a]$; in degree d there is in fact $ad + 1$ of them, so that I is in fact the saturated ideal of the curve. □

Exercise 1.184. ?? In order to do the same as we did in the previous exercise for surface scrolls, prove that the Hilbert polynomial $f_S(d)$ of the surface scroll $S(a, b) \subset \mathbb{P}^{a+b+1}$ satisfies

$$f_S(d) \geq (a+b)\binom{d+1}{2} + d + 1.$$

Solution to Exercise ??: We know that the intersection of S with a general hyperplane H is a rational normal curve C of degree $a + b$; this is because it intersect every fiber of the scroll once, hence is rational, has degree $a + b$ because that is the degree of S, and it is nondegenerate because H is general. Then, calling $f_S(d)$ and $f_C(d)$ the Hilbert functions of respectively S and C, we have the inequality

$$f_S(d) - f_S(d-1) \geq f_C(d)$$

that comes from the sequence

$$0 \to k[x_0, \dots, x_{a+b+1}]/I_S \xrightarrow{h=\cdot H} k[x_0, \dots, x_{a+b+1}]/I_S \xrightarrow{p} k[x_0, \dots, x_{a+b+1}]/I_C$$

where h is injective and $p \circ h = 0$. Then, we have $f_C(d) = (a+b)d + 1$, and from the inequality above we get

$$f_S(d) \geq \sum_{e=0}^{d} f_C(e) = (a+b)\binom{d+1}{2} + d + 1$$

as requested. Note that using the same argument it is possible to prove that surface scrolls are nondegenerate surfaces with *minimal Hilbert function*; for curves, the same is true for rational normal curves. □

Exercise 1.185. ?? Let $x_0, \dots, x_{a+b+1}$ be coordinates in $\mathbb{P}^{a+b+1}$. Prove that the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} & x_{a+1} & x_{a+2} & \dots & x_{a+b} \\ x_1 & x_2 & \dots & x_a & x_{a+2} & x_{a+3} & \dots & x_{a+b+1} \end{pmatrix}$$

vanish on a surface scroll $S(a, b)$. As in Exercise **??**, show that the ideal I generated by the minors defines the surface scheme theoretically. Then, using Exercise **??**, prove that I is the saturated ideal of the surface scroll.

Solution to Exercise ??: Solving the equations of the minors, it is easy to see that set theoretically the vanishing locus is the union of the lines

$$L_{[s,t]} = \overline{[s^a, s^{a-1}t, \dots, t^a, 0, \dots, 0], [0, \dots, 0, s^b, s^{b-1}t, \dots, t^b]}$$

so it is a surface scroll; it is composed of lines joining two rational normal curves of degree a and b, so it is of type $S(a,b)$. Using an argument as in Exercise **??**, it is possible to prove that the tangent space of the vanishing locus of the minors has a 2 dimensional tangent space, so that it is $S(a,b)$ also scheme theoretically. A basis of monomials for $K[x_0, x_1, \ldots, x_{a+b+1}]/I$ is the set of all monomials of the kind $x_i^e x_{i+1}^f x_{a+1}^g$ where $0 \leq i \leq a$ or of the kind $x_a^e x_j^f x_{j+1}^g$ where $a \leq j \leq a+b$; it is not hard to prove that they are a basis for the coordinate ring, and that there is exactly $(a+b)\binom{d+1}{2} + d + 1$ many of them. This proves that the ideal is saturated, and that in fact the inequality of the previous exercise is an equality. □

Exercise 1.186. **??** Let X be a smooth projective variety, $\mathcal{E}$ a vector bundle on X and $\mathbb{P}E$ its projectivization. Let L be any line bundle on X; as we've seen, there is a natural isomorphism $\mathbb{P}\mathcal{E} \cong \mathbb{P}(E \otimes L)$.

(a) How does the class $c_1(\mathcal{O}_{\mathbb{P}(E\otimes L)}(1))$ relate to $c_1(\mathcal{O}_{\mathbb{P}E}(1))$?
(b) Using the results of Section **??**, show that the two descriptions of the Chow ring of $\mathbb{P}E = \mathbb{P}(E \otimes L)$ agree.

Solution to Exercise ??: We have $\mathcal{O}_{\mathbb{P}(E\otimes L)}(-1)$ sitting inside $\pi^*(E \otimes L)$ as tautological bundle; this means that it is equal to $\mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^* L$; dualizing, we get

$$c_1(\mathcal{O}_{\mathbb{P}(E\otimes L)}(1)) = c_1(\mathcal{O}_{\mathbb{P}E}(1)) - \pi^* c_1(L).$$

Let us give names: let us denote by ζ_0 and ζ_1 the two classes $c_1(\mathcal{O}_{\mathbb{P}E}(1))$ and $c_1(\mathcal{O}_{\mathbb{P}(E\otimes L)}(1))$, and by λ the class $\pi^* c_1(L)$; then, let us call $c_1, \ldots, c_{r+1}$ the (pullbacks of) Chern classes of E. Now, the Chow ring of $\mathbb{P}(E)$ is

$$A^*(\mathbb{P}(E)) = A^*(X)[\zeta_0]/(\zeta_0^{r+1} + c_1\zeta_0^r + \ldots + c_{r+1})$$

using $\zeta = \zeta' + \lambda$ that proved above, we get

$$A^*(\mathbb{P}(E)) = A^*(X)[\zeta_1]/(\zeta_1^{r+1} + (c_1 + (r+1)\lambda)\zeta_1^r + \ldots + (\lambda^{r+1} + \ldots + c_{r+1}))$$

and as coefficients of ζ_1^i we find, as from Section **??**, Chern classes of $E \otimes L$; so the two descriptions of the Chow ring agree. □

Exercise 1.187. **??** Let $\pi : Y \to X$ be a projective bundle.

(a) Show that the direct sum decomposition of the group $A(X)$ given in Theorem **??** *depends* on the choice of vector bundle $\mathcal{E}$ with $Y \cong \mathbb{P}\mathcal{E}$.
(b) Show that if we define group homomorphisms $\psi_i : A(Y) \to A(X)^{\oplus i+1}$ by

$$\psi_i : \alpha \mapsto \big(\pi_*(\alpha), \pi_*(\zeta\alpha), \ldots, \pi_*(\zeta^i\alpha)\big)$$

then the filtration of $A(Y)$ given by

$$A(Y) \supset \mathrm{Ker}(\psi_0) \supset \mathrm{Ker}(\psi_1) \supset \cdots \supset \mathrm{Ker}(\psi_{r-1}) \supset \mathrm{Ker}(\psi_r) = 0$$

is *independent* of the choice of $\mathcal{E}$. (Hint: give a geometric characterization of the cycles in each subspace of $A(Y)$.)

Solution to Exercise ??: The very definition of the direct sum decomposition relies on ζ, that depends on the choice of E. The coordinates for this decomposition are the coefficients of powers of ζ, and the automorphism sending ζ to $\zeta' + \lambda$ (that we have seen in the previous exercise happens when we change the representative E) will of course not keep the decomposition, because it does not act diagonally; under this point of view, it acts as an upper triangular transformation (it is possible to prove (b) using this fact, but we will use the hint instead). For part (b), remember that $\pi_*(\alpha) = 0$ if and only if π on α has positive dimensional fibers; so, $\pi(\alpha) = \ldots = \pi_*(\zeta^i \alpha) = 0$ if and only if π on α has fibers of dimension $> i$ (because ζ is generically an hyperplane section of fibers); this gives a characterization of $\mathrm{Ker}(\psi_i)$ that does not depend on ζ, and hence on the representative E.

meetsin this way, it is easy to see that $\mathrm{Ker}(\psi_i)$ is composed by all cycles whose intersection □

Exercise 1.188. **??** Show that the product of a base point free linear series with a very ample linear series is very ample. (Hint: prove that the product separates points and tangent vectors.)

Solution to Exercise ??: This is a simple application of Lemma **??**, where we take π to be the identity; in fact, being relatively ample for the identity map is the same as being base point free. □

Exercise 1.189. **??** Let $\mathcal{F}$ be a vector bundle of rank r on a scheme X. Prove that $\mathcal{F}$ is very ample if and only if, for each finite subscheme $Y \subset X$ of length 2 we have

$$\dim H^0(\mathcal{F}(-Y)) \leq \dim H^0(\mathcal{F}) - 2r, \text{ (in which case equality holds)},$$

where $H^0(\mathcal{F}(-Y))$ denotes the space of sections of $\mathcal{F}$ that vanish on Y (the section of the kernel of the map $\mathcal{F} \to \mathcal{F}/\mathcal{I}_{Y/X}\mathcal{F}$.

Solution to Exercise ??: Let $\mathcal{E} = \mathcal{F}^*$, and let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection. Since every subscheme of length 2 in $\mathbb{P}(\mathcal{E})$ is contained in the preimage $\pi^{-1}Y = \mathbb{P}(\mathcal{E}|_Y)$ of some subscheme Y of length 2, the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ is very ample if and only if its restriction to such preimages is very ample. Since every vector bundle on a finite scheme is trivial we have $H^0(\mathcal{F}|_Y) = \mathcal{O}_Y^r$, so $\mathbb{P}\mathcal{E}_Y = Y \times \mathbb{P}^{r-1}$, and the restriction of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ to this scheme is $\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)$.

The complete linear series $|\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)|$ is very ample(for example by Proposition **??** where we take both $\mathcal{L}$ and $\mathcal{F}$ to be trivial bundles), and has dimension $2r$. Thus $\dim H^0(\mathcal{F}(-Y)) \leq \dim H^0(\mathcal{F}) - 2r$ if and only if the map $H^0(\mathcal{F}) \to H^0(\mathcal{F}|_Y)$ is surjective (in which case the inequality is an equality.) It thus suffices to show that no linear series of the form $\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1), W$, with $W \subsetneq H^0\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)$, can be very ample on $Y \times \mathbb{P}^{r-1}$. Equivalently: there is no embedding of $Y \times \mathbb{P}^{r-1}$ in $\mathbb{P}^{2r-2}$ taking $Y_{red} \times \mathbb{P}^{r-1}$ to the disjoint union of (one or two) linear spaces.

Suppose, on the contrary, there were such an embedding. Since we assume that the ground field K is algebraically closed, there are only two possibilities for Y: two reduced points or one double point. If Y consists of two reduced points, so that the image of $Y_{red} \times \mathbb{P}^{r-1}$ is the union of two linear spaces, we get a contradiction simply from the fact that any two $r-1$-dimensional varieties in $\mathbb{P}^r$ meet.

On the other hand, if Y is a double point, $Y = \operatorname{Spec} K[\epsilon]/(\epsilon^2)$, then the normal bundle of Y_{red} is Y is trivial, and thus the normal bundle of $Y_{\text{red}} \times \mathbb{P}^{r-1}$ in $Y \times \mathbb{P}^{r-1}$ is trivial. This bundle is a subbundle of the normal bundle of the linear space $Y_{\text{red}} \times \mathbb{P}^{r-1}$ in $\mathbb{P}^{2r-2}$, so the top Chern class of this bundle is trivial. However, this normal bundle is isomorphic to $\mathcal{O}(1)^{r-2}$, and the degree of its top Chern class is actually 1, contradicting the existence of the embedding. (We could state this argument without mentioning Chern classes: the content is that any $r-2$ linear forms on $\mathbb{P}^{r-1}$ have a common zero.) $\square$

Exercise 1.190 (Vector Bundles on Elliptic Curves)**. ??** We will apply the criterion of Exercise **??** to prove the existence of certain embeddings of a scroll over an elliptic curve in Exercise **??**. To do this we need some facts about vector bundles on an elliptic curve from the rather complete theory of **?**]. We invite the reader to prove what we will need:

(a) Show that there is a unique indecomposable vectore bundle $\mathcal{E} := \mathcal{E}(2, \mathcal{L})$ of rank 2 on the genus 1 curve C with given determinant $\mathcal{L} := \wedge^2\mathcal{E}$ of degree 1, and that there is a short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0.$$

Similarly, show that there is a unique indecomposable rank 2 bundle with determinant $\mathcal{O}_C$.

(b) Deduce that there is a unique indecomposable bundle $\mathcal{E}(2, \mathcal{L})$ of rank 2 with any given determinant $\mathcal{L}$, and that for any line bundle $\mathcal{L}'$ of degree $\lfloor \deg \mathcal{L}/2 \rfloor$ there is an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E}(2, \mathcal{L}) \to \mathcal{L} \otimes \mathcal{L}'^{-1} \to 0.$$

If $d > 0$ then $h^0\mathcal{E}(2, \mathcal{L}) = d$ and $h^1\mathcal{E}(2, \mathcal{L}) = 0$.

Solution to Exercise ??: (a) Since $\operatorname{Ext}^1(\mathcal{L}, \mathcal{O}_C) = H^1(\mathcal{L}^{-1})$ is 1-dimensional, there is an extension of the form given, and the bundle $\mathcal{E}$ in the middle is unique up to isomorphism. We define $\mathcal{E}(2, \mathcal{L})$ to be this bundle.

Suppose that $\mathcal{E}(2, \mathcal{L})$ were decomposable; say $\mathcal{E}(2, \mathcal{L}) = \mathcal{L}' \oplus \mathcal{L}''$, with $\mathcal{L} = \mathcal{L}' \otimes \mathcal{L}''$. Since $\deg \mathcal{L}' + \deg \mathcal{L}'' = 1$, at least one of the two bundles has degree

≥ 1; suppose this is true of $\mathcal{L}'$. From the exact sequence defining $\mathcal{E}(2,\mathcal{L})$ we get an exact sequence

$$0 \to Hom(\mathcal{L}', \mathcal{O}_C) \to Hom(\mathcal{L}', \mathcal{E}(2,\mathcal{L})) \to Hom(\mathcal{L}', \mathcal{L}) \to \cdots .$$

The first term vanishes for degree reasons, so the inclusion map induces a nonzero homomorphism $\mathcal{L}' \to \mathcal{L}$. Since $\deg \mathcal{L}' \geq \deg \mathcal{L}$ it follows that $\mathcal{L}' \cong \mathcal{L}$. But this implies that the sequence defining $\mathcal{E}(2,\mathcal{L})$ is split, a contradiction.

Conversely, given an indecomposable rank 2 bundle $\mathcal{E}$ with determinant $\mathcal{L}$ of degree 1, The Riemann-Roch formula for vector bundles on a curve gives

$$h^0\mathcal{E} - h^1\mathcal{E} = \deg \mathcal{E} - (\operatorname{rank} \mathcal{E})(1-g) = 1,$$

so $\mathcal{E}$ has at least one global section, σ. Let $\mathcal{L}'$ be the preimage in $\mathcal{E}$ of the torsion in $\mathcal{E}/\mathcal{O}_C\sigma$. Since both $\mathcal{L}'' := \mathcal{E}/\mathcal{L}'$ and $\mathcal{L}'$ are torsion free, they are line bundles on C, and we have a short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E} \to \mathcal{L}'' \to 0.$$

Taking determinants, we see that $\mathcal{L}'' = \mathcal{L} \otimes \mathcal{L}'^{-1}$. Since $H^0\mathcal{L}'$ contains the section σ, we have $\deg \mathcal{L}' \geq 0$, and $\deg \mathcal{L}'' = \deg \mathcal{L} - \deg \mathcal{L}' = 1 - \deg \mathcal{L}'$. Since we supposed $\mathcal{E}$ indecomposable, it follows that the sequence above is not split; that is,

$$\operatorname{Ext}^1(\mathcal{L}', \mathcal{L}'') = H^1(\mathcal{L}' \otimes \mathcal{L}'^{-1}) \neq 0.$$

Since the degree of $\mathcal{L}' \otimes \mathcal{L}'^{-1}$ is $\deg \mathcal{L}' - (1 - \deg \mathcal{L}') = 2\deg \mathcal{L}' - 1$, we must have $\deg \mathcal{L}' = 0$, so $\mathcal{L}' = \mathcal{O}_C$ and $\mathcal{L}'' = \mathcal{L}$ as required. Thus $\mathcal{E} = \mathcal{E}(2,\mathcal{L})$.

(b) Let $\mathcal{L}'$ be a bundle of degree e, and set $\mathcal{L}_0 = \mathcal{L} \otimes \mathcal{L}'^{-2}$, a bundle of degree 1. Set $\mathcal{E}(2,d) = \mathcal{L}' \otimes \mathcal{E}(2,\mathcal{L}_0)$, which is indecomposable, has rank 2 and determinant $\mathcal{L}$. Given any other such bundle, the tensor product with $\mathcal{L}'^{-1}$ would give another indecomposable bundle of rank 2 and determinant $\mathcal{L}_0$; since we have shown that such a bundle is unique up to isomorphism, it follows that $\mathcal{E}(2,\mathcal{L})$ is the only indecomposable bundle with this rank and determinant. From the definitions there is a short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E}(2,\mathcal{L}) \to \mathcal{L} \otimes \mathcal{L}'^{-1} \to 0.$$

In case $d \geq 2$ the values for the $h^i\mathcal{E}(2,\mathcal{L})$ follow at once from the associated long exact sequence, using $H^1\mathcal{L}' = 0 = H^1\mathcal{L} \otimes \mathcal{L}'^{-1}$. If $d = 1$ the we tensor the defining sequence for $\mathcal{E}(2,\mathcal{L})$ with a nontrivial line bundle of degree 0 to get a sequence

$$0 \to \mathcal{L}' \to \mathcal{E}(2,\mathcal{L}) \to \mathcal{L} \otimes \mathcal{L}'^{-1} \to 0.$$

Since $H^0\mathcal{L}' = 0 = H^1\mathcal{L}'$, the desired result follows at once from the long exact sequence. (Using similar ideas, one can show that $h^0(\mathcal{E}(2,\mathcal{L}) = h^1(\mathcal{E}(2,\mathcal{L})) = 1$ when $\mathcal{L}$ has degree 0.)

□

Exercise 1.191 (Elliptic Scrolls)**.** **??** Let $\mathcal{L}$ be a line bundle of degree d on an elliptic curve C, and let $\mathcal{E} = \mathcal{E}(2, \mathcal{L})^*$. Show that $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is very ample if and only if $d \geq 5$. In particular, taking $d = 5$, deduce that there is an embedding of $\mathbb{P}\mathcal{E}$ in $\mathbb{P}^4$ as a smooth surface of degree 5, an *elliptic quintic scroll*. (Hint: If $d \geq 5$ then for every divisor D of degree 2 on C, we have $h^0(\mathcal{E}(2, \mathcal{L})(-D)) = d - 2 * \operatorname{rank} \mathcal{E} = d - 4$ by Exercise **??**, so we can use the criterion of Exercise **??**. On the other hand, if $d = 4$ or less then there is some line bundle of degree 2 such that $h^0(\mathcal{E}(2, \mathcal{L})(-D)) \neq d - 4$.

One can decompose such a surface geometrically as follows. From Exercise **??** it follows that every line bundle of degree -3 can be embedded in $\mathcal{E}$, and gives rise to a section of the scroll. Since $\deg \mathcal{E} = 5 = 3 + 3 - 1$, any two of these sections, say X_1, X_2 meet in a single point p, and in fact lie in planes in $\mathbb{P}^4$ that meet only at p. The scroll S is the closure of the union of the lines connecting corresponding points of these two sections other than p. For more information see **?**] and **?**].

The elliptic quintic scrolls are closely related to the Horrocks-Mumford bundle on $\mathbb{P}^4$; see for example **?**]. The homogeneous coordinate rings of the scrolls are normal domains, and even nonsingular in codimension 2, but not Cohen-Macaulay (because H^1 of their structure sheaves is nonzero); in many ways they are the simplest examples of such rings.

Solution to Exercise ??: By Exercise **??**, as soon as $d \geq 5$, for every Y scheme of degree 2 we have $h^0(\mathcal{E}(2, \mathcal{L})(-D)) = d - 4$ and $h^0(\mathcal{E}(2, \mathcal{L})) = d$ that proves the very ampleness. So, for $d = 5$, the embedding from the linear series $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ is an embedding as a surface (of degree 5) in $\mathbb{P}^4$, ruled by lines over an elliptic curve basis. □

Exercise 1.192. **??** In Example **??** we used intersection theory to show that there does not a exist a rational solution to the general quadratic polynomial, that is, there do not exist rational functions $X(a, \ldots, f)$, $Y(a, \ldots, f)$ and $Z(a, \ldots, f)$ such that

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \equiv 0.$$

To gain some appreciation of the usefulness of intersection theory, give an elementary proof of this assertion.

Solution to Exercise ??: Suppose there exist a rational section $f : \mathbb{P}^5 - \to \mathbb{P}^2 \times \mathbb{P}^5$, and let us take its restriction on the $\mathbb{P}^2$ given by $d = e = f = 0$. It is possible that f is not defined at all on this $\mathbb{P}^2$; if this is the case, we can make a coordinate change on X, Y and Z (and correspondingly on a, b, c, d, e, f) in such a way this does not happen anymore; it is important to notice that this is true because the net $aX^2 + bY^2 + cZ^2$ sweeps out the whole $\mathbb{P}^5$ under changes of coordinates. Clearing denominators, we can assume X, Y, Z to be homogeneous polynomials in a, b, c without common factors, so we need to find polynomials satisfying

$$aX(a,b,c)^2 + bY(a,b,c)^2 + cZ(a,b,c)^2 = 0.$$

Setting $c = 0$, we get the equation $aX(a,b,0)^2 + bY(a,b,0)^2 = 0$; comparing parity of roots, we get $X(a,b,0) = Y(a,b,0) = 0$, that means c is a factor of X and Y; in the same way, we get that X is a multiple of bc, Y is a multile of ac, and Z a multiple of ab. Putting all together in our equation, we get

$$ab^2c^2\left(\frac{X(a,b,c)}{bc}\right)^2 + a^2bc^2\left(\frac{Y(a,b,c)}{ac}\right)^2 + a^2b^2c\left(\frac{Z(a,b,c)}{ab}\right)^2 = 0,$$

that in turn proves that a is a factor of X, but this is impossible because a is already a factor of Y and Z, and they were coprime polynomials. □

Exercise 1.193. ?? Let

$$\Phi = \left\{(L,p) \in \mathbb{G}(1,n) \times \mathbb{P}^n \mid p \in L\right\}$$

be the universal line in $\mathbb{P}^n$, and let σ_1, $\sigma_{1,1}$ and ζ be the pullbacks of the Schubert classes $\sigma_1 \in A^1(\mathbb{G}(1,n))$, $\sigma_{1,1} \in A^2(\mathbb{G}(1,n))$ and the hyperplane class $\zeta \in A^1(\mathbb{P}^n)$ respectively. Find the degree of all monomials $\sigma_1^a\sigma_{1,1}^b\zeta^c$ of top degree $a + 2b + c = \dim(\Phi) = 2n - 1$.

Solution to Exercise ??: We can look at Φ as $\mathbb{P}(S)$ on $\mathbb{G}(1,n)$; with this identification, we have $\zeta = c_1(\mathcal{O}_{\mathbb{P}(S)}(1))$, because an hyperplane section of $\mathbb{P}^n$ can be seen as a section of S^* on $\mathbb{G}(1,3)$. In this way, we get the relation

$$\zeta^2 = \zeta\sigma_1 - \sigma_{1,1}.$$

Iterating this formula (by induction), we get

$$\zeta^c = \sigma_{c-1}\zeta - \sigma_{c-1,1}$$

Now, top degree classes where ζ do not appear are going to be zero, because they are on $\mathbb{G}(1,n)$. Then whenever τ is a zero dimensional class on $\mathbb{G}(1,3)$, we of course going to have

$$\deg_\Phi(\tau\zeta) = \deg_{\mathbb{G}(1,n)}(\tau)$$

Collecting everything, we get

$$\deg_\Phi(\sigma_1^a\sigma_{1,1}^b\zeta^c) = \deg_{\mathbb{G}(1,n)}(\sigma_1^a\sigma_{1,1}^b\sigma_{c-1})$$

that moves the problem to a completely combinatoric one involving only the Chow ring of $\mathbb{G}(1,n)$; it can be solved for instance using the hook formula (see Exercise **??**) from which we get

$$\deg_\Phi(\sigma_1^a\sigma_{1,1}^b\zeta^c) = \frac{1}{a+1}\binom{a+1}{n-1-b}$$

if $a \geq c-1 \geq 0$, and zero otherwise. □

Exercise 1.194. **??** Consider the flag variety of pairs consisting of a point $p \in \mathbb{P}^3$ and a line $L \subset \mathbb{P}^3$ containing p; that is,

$$\mathbb{F} = \{(p,L) \in \mathbb{P}^3 \times \mathbb{G}(1,3) \mid p \in L \subset \mathbb{P}^3\}.$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^1$-bundle over $\mathbb{G}(1,3)$, or as a $\mathbb{P}^2$-bundle over $\mathbb{P}^3$. Calculate the Chow ring $A(\mathbb{F})$ via each map, and show that the two descriptions agree.

Solution to Exercise ??: We have $\mathbb{F} = \mathbb{P}_{\mathbb{G}_{1,3}}(S)$, so that we get

$$A*(\mathbb{F}) = A*(\mathbb{G}(1,3))[\zeta_0]/(\zeta_0^2 - \sigma_1\zeta_0 + \sigma_{1,1}).$$

On $\mathbb{P}^3$ (with hyperplane class η), the space $\mathbb{F}$ can be seen as $\mathbb{P}(Q)$; we then have

$$A*(\mathbb{F}) = A*(\mathbb{P}^3)[\zeta_1]/(\zeta_1^3 + \eta\zeta_1^2 + \eta^2\zeta_1 + \eta^3).$$

We can see that this two are isomorphic by a ring morphism sending ζ_0 to η (they both represent the class of flag with the point belonging to an hyperplane) and σ_1 to $\zeta_1 + \eta$ (they both are the classes of flags where the line meets a given line). □

Exercise 1.195. **??** By Theorem **??**, the Chow ring of the product $\mathbb{P}^3 \times \mathbb{G}(1,3)$ is just the tensor product of their Chow rings; that is

$$A(\mathbb{P}^3 \times \mathbb{G}(1,3)) = A(\mathbb{G}(1,3))[\zeta]/(\zeta^4).$$

In these terms, find the class of the flag variety $\mathbb{F} \subset \mathbb{P}^3 \times \mathbb{G}(1,3)$ of Exercise **??**.

Solution to Exercise ??: The class is codimension 2, so its class is going to be of the form

$$[\mathbb{F}] = \alpha\zeta^2 + \beta\zeta\sigma_1 + \gamma\sigma_2 + \delta\sigma_{1,1}.$$

To find coefficients, we need to intersect with dual classes (respectively) $\zeta^2\sigma_{2,2}$, $\zeta^3\sigma_{2,1}$, $\zeta^4\sigma_2$, $\zeta^4\sigma_{1,1}$. A cycle in the class $\zeta^2\sigma_{2,2}$ is composed by couples where the line is fixed and the point is in a general plane, so that we have 1 point of $\mathbb{F}$, that means, $\alpha = 1$. A cycle in the class $\zeta^3\sigma_{2,1}$ has lines in a general pencil (contained in a plane and through a fixed point) and the point in a general line, that gives $\beta = 1$. A cycle in the class $\zeta^4\sigma_2$ has lines through a general point and the point fixed, that gives $\gamma = 1$. A cycle in the class $\zeta^4\sigma_{1,1}$ has lines contained in a general plane and the point fixed, that gives $\delta = 0$. The class is then

$$[\mathbb{F}] = \zeta^2 + \zeta\sigma_1 + \sigma_2.$$

This can be seen also from the fact that this arises as the vanishing locus of the tautological morphism of vector bundles $\pi_1^*\mathcal{O}_{\mathbb{P}^3}(-1) \to \pi_2^*\mathcal{Q}$, that means, as vanishing locus of a section of the vector bundle $\pi_1^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes \pi_2^*\mathcal{Q}$, whose second Chern class is the one above. Check of transversality can be made through Kleiman's transversality theorem, or by an explicit evaluation of tangent spaces. □

Exercise 1.196. **??** Generalizing the preceding problem, let

$$\mathbb{F}(0,k,r) = \{(p,\Lambda) \in \mathbb{P}^r \times \mathbb{G}(k,r) \mid p \in \Lambda \subset \mathbb{P}^r\}.$$

Find the class of $\mathbb{F}(0,1,r) \subset \mathbb{P}^r \times \mathbb{G}(k,r)$.

Solution to Exercise ??: As in the previous exercise, given that $\mathbb{F}(0,k,r)$ has codimension $k-r$ in $\mathbb{P}^r \times \mathbb{G}(k,r)$, we have

$$[\mathbb{F}] = \sum_{i+|\lambda|=k-r} \alpha_\lambda \zeta^i \sigma_\lambda$$

and we can find the coefficient α_λ intersecting with the dual class $\zeta^{r-i}\sigma_{\lambda^*}$, that means, the k plane is in a σ_{λ^*} cycle, and the point in a $\mathbb{P}^i$; we need then to know how many planes in a cycle σ_{λ^*} intersect a general $\mathbb{P}^i$, that is the same as the degree of $\sigma_{\lambda^*}\sigma_{r-k-i}$, that is $\sigma_{\lambda^*}\sigma_{|\lambda|}$. Note that this cycle has the right codimension, and it is nonzero if and only if σ_λ is σ_λ (that means, only if the Young diagram has only one row). This gives us

$$[\mathbb{F}] = \sum_{j=0}^{r-k} \zeta^{r-k-j}\sigma_j$$

that again could be recovered as section of the vector bundle $\pi_1^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \pi_2^*\mathcal{Q}$ whose top Chern class is the above. □

Exercise 1.197. **??** Generalizing Exercise **??** in a different direction, let

$$\Phi_r = \{(L,M) \in \mathbb{G}(1,r) \times \mathbb{G}(1,r) \mid L \cap M \neq \varnothing\}.$$

Given that, by Theorem **??** we have

$$A\big(\mathbb{G}(1,r) \times \mathbb{G}(1,r)\big) \cong A\big(\mathbb{G}(1,r)\big) \otimes A\big(\mathbb{G}(1,r)\big),$$

find the class of Φ_r in $A\big(\mathbb{G}(1,r) \times \mathbb{G}(1,r)\big)$

(a) $r = 3$;
(b) $r = 4$; and
(c) for general r.

Solution to Exercise ??: For $n = 3$, the codimension of Φ_3 is one, so it will be a linear combination of σ_1 and σ_1'; to find the coefficients, we need to intersect with a pencil

$\sigma_{2,1}\sigma'_{2,2}$ of couples with the first line in a pencil, the second line fixed: this intersects Φ_3 once, so that we get

$$[\Phi_3] = \sigma_1 + \sigma'_1.$$

For $n = 4$, the codimension is two, so that we have classes σ_2, $\sigma_{1,1}$, $\sigma_1\sigma'_1$,σ'_2 and $\sigma'_{1,1}$. It is easy to check, again intersecting with dual cycles, that coefficients of σ_2 and σ'_2 are 1, and coefficients of $\sigma_{1,1}$ and $\sigma'_{1,1}$ are 0. To find the coefficient of $\sigma_1\sigma'_1$, we need to intersect with the class $\sigma_{2,1}\sigma'_{2,1}$, that means, couples of lines where both are free to move in a pencil; the two planes these pencils span will intersect in one point, so we get one element of Φ_4 here as well. We then got

$$[\Phi_4] = \sigma_2 + \sigma_1\sigma'_1 + \sigma'_2.$$

In the general case, Φ_r will have codimension $r - 2$ in $\mathbb{G}(1, r) \times \mathbb{G}(1, r)$; the class will be of the kind

$$[\Phi_r] = \sum_{|\lambda|+|\lambda'|=r-2} \alpha_{\lambda,\lambda'}\sigma_\lambda\sigma'_{\lambda'}.$$

To find coefficients we need to intersect with classes $\sigma_{\lambda*}\sigma'_{(\lambda')*}$; in order for this class to meet Φ_r, we need the total space that lines in a general $\sigma_{\lambda*}$ cycle span too meet the total space of lines in a general $\sigma'_{(\lambda')*}$ cycle. Counting dimensions, this happens if and only if both Young diagrams λ and λ' have only one row, and in that case it is immediate to see that the coefficient is one. We then get

$$[\Phi_r] = \sum_{i+j=r-2} \sigma_i\sigma'_j.$$

This classes are also related to vector bundles; in particular, considering the tautological vector bundle morphism $\pi_1^*S \to \pi_2^*Q$, this is the locus where this morphism fails to have rank 2 (not the same thing as the vanishing locus!). The technique to deal with this kind of problems will be the main content of Chapter **??**, and we will solve this problem again in Exercise **??**. □

Exercise 1.198. **??** Let Z be the blowup of $\mathbb{P}^n$ along an $(r - 1)$-plane, and let $E \subset Z$ be the exceptional divisor. Find the degree of the top power $e^n \in A(Z)$.

Solution to Exercise ??: Following notation in Corollary **??**, we need to find the degree of $(\zeta - \alpha)^n$. Using the equalities in Corollary **??**, we get that the degree of $\zeta^i\alpha^{n-i}$ is zero whenever $r \geq i + 1$, and 1 otherwise. We then get as degree

$$1 - n + \binom{n}{2} - \ldots + (-1)^{n-r}\binom{n}{n-r}$$

that for instance agrees with -1 in the case of a point in $\mathbb{P}^2$. □

Exercise 1.199. **??** Again let $Z = Bl_\Lambda\mathbb{P}^n$ be the blowup of $\mathbb{P}^n$ along an $(r-1)$-plane Λ. In terms of the description of the Chow ring of Z given in Corollary **??**, find the classes of the following:

(a) the proper transform of a linear space $\mathbb{P}^s$ containing Λ, for each $s > r$;
(b) the proper transform of a linear space $\mathbb{P}^s$ in general position with respect to Λ (that is, disjoint from Λ if $s \leq n-r$; and transverse to Λ if $s > n-r$); and
(c) in general, the proper transform of a linear space $\mathbb{P}^s$ intersecting Λ is an l-plane.

Solution to Exercise ??: Remember that the class ζ is the class of an hyperplane in $\mathbb{P}^n$ that is transverse to Λ, and α is the class of the proper transform of an hyperplane containig Λ; for the first point, such $\mathbb{P}^s$ is obtained as transverse intersection of $n-s$ proper transforms of hyperplanes containing Λ; the class is then α^{n-s}. In the second case, for the same reason the class is ζ^{n-s}. In the third case, the class will be

$$\zeta^{r-1-l}\alpha^{n-s-(r-1-l)}$$

because such a $\mathbb{P}^s$ is the intersection of $r-1-l$ hyperplanes of type ζ (that decrease the dimension of the intersection with Λ down to l) and $n-s-(r-1-l)$ hyperplanes of type α. □

Exercise 1.200. **??** Let $Z = Bl_L\mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along a line. In terms of the description of the Chow ring of Z given in Corollary **??**, find the classes of the proper transform of a smooth surface $S \subset \mathbb{P}^3$ of degree d containing L.

Solution to Exercise ??: The class of the proper transform $\tilde{S}$ will be of the kind $u\zeta + v\alpha$; to find the coefficients u and v, we need to intersect with dual classes $\zeta\alpha$ and $\zeta^2 - \zeta\alpha$, respectively of the proper transform of a line meeting L in a point, and of a fiber of E over a point of L. In the first case the intersection is $d-1$ points, because the (transverse) point of intersection of S and the cycle along L goes away in the blow up; for the second intersection, S is smooth along L, so locally around L the surface S will look like the embedding

$$\mathcal{N}_{L/S} \hookrightarrow \mathcal{N}_{L/\mathbb{P}^3}$$

that when we go to the projectivization it will look like a curve in $\mathbb{P}^1 \times \mathbb{P}^1$, meeting every fiber once. So, collecting everything, we get

$$[\tilde{S}] = (d-1)\zeta + \alpha = d\zeta - E$$

that could be found also observing that the pullback of S contains the exceptional divisor with multiplicity 1. □

Exercise 1.201. **??** Now let $Z = Bl_L\mathbb{P}^4$ be the blowup of $\mathbb{P}^4$ along a line, and let $S \subset \mathbb{P}^4$ be a smooth surface of degree d containing L. Show by example that the class of the proper transform of S in Z is not determined by this data. For example, try taking S a cubic scroll, with L either

(a) a line of the ruling of S; or
(b) the directrix of S

and seeing that you get different answers.

Solution to Exercise ??: The class we are looking for will be of the kind

$$[\tilde{S}] = u\zeta^2 + v\zeta\alpha + w\alpha^2.$$

To find the coefficients, we will need to intersect with dual classes; remembering that degrees of α^4 and $\zeta\alpha^3$ are zero, and the degrees of $\zeta^2\alpha^2, \zeta^3\alpha$ and ζ^4 is one, we get that dual classes are (respectively) $\alpha^2, \zeta\alpha - \alpha^2$ and $\zeta^2 - \zeta\alpha$. The class α^2 is the class of the proper transform of a $\mathbb{P}^2$ containing L; intersection with $\tilde{S}$ will happen only outside the exceptional divisor E, hence the intersection number (and so u) will be the number of points of intersection between S and a plane containing L away from L. The class $\zeta\alpha - \alpha^2 = \alpha \cdot E$ is the intersection of the exceptional divisor and of the proper transform of an hyperplane containing L; so, we consider the intersection of S and an hyperplane H containing L (that will be a curve C together with L itself), and then we consider the proper transform $\tilde{C}$ of C in $Bl_L H$, and in particular in how many points $\tilde{C}$ intersects the exceptional divisor; this goes back just to the number of times C and L intersect, that will be the coefficient v. Similarly, the class $\zeta^2 - \zeta\alpha = \zeta \cdot E$ is the intersection of a general hyperplane, and the exceptional divisor; we consider then the intersection D of S with a general hyperplane H (that intersects L at a point p, that belongs to D too) and we ask for the intersection of the proper transform $\tilde{D}$ of D in $Bl_p H$ and the exceptional divisor; this number (and hence, w) is always 1, as soon as S is smooth along L.

In the first example, a general hyperplane section of S containing the directrix is L together with two lines M_1 and M_2 of the ruling, each meeting L at one point; we then immediately get $v = 2$. Intersecting with another plane containing L, the intersection will be only supported at L, so we get $u = 0$: we then get

$$[\tilde{S}] = 2\zeta\alpha + \alpha^2.$$

In the second example, the intersection with a general hyperplane containing L will contain also a conic C touching every line of the ruling (hence, L) once; this will give us $v = 1$ and, intersecting with another hyperplane containing L we get one intersection point outside L, that means,

$$[\tilde{S}] = \zeta^2 + \zeta\alpha + \alpha^2.$$

The class then does not depend only on the degree of d. More specificly, suppose we have $\mathcal{N}_{L/S} = \mathcal{O}_L(e)$; calling H the hyperplane section on S, we have $deg(L^2) = e$, $deg(L \cdot H) = 1$ and of course $deg(H^2) = d$. Remembering that $C = H - L$, we get

$$v = L \cdot C = L \cdot (H - L) = 1 - e$$

$$u = C \cdot C = (H - L)^2 = d - 2 + e$$

that in fact agrees with out previous examples, in which, respectively, $e = -1$ and $e = 0$ (note that the sum of the three coefficients is always d). □

Exercise 1.202. **??** Let $Z = Bl_\Lambda \mathbb{P}^n$ be the blowup of $\mathbb{P}^n$ along an $(r-1)$-plane Λ; that is, if we consider the subspace $\mathbb{P}^{n-r} \subset \mathbb{G}(r,n)$ of r-planes containing Λ, we have

$$Z = \{(p, \Gamma) \in \mathbb{P}^n \times \mathbb{P}^{n-r} \mid p \in \Gamma\}.$$

Using the description of the Chow ring of Z given in Corollary **??**, find the class of $Z \subset \mathbb{P}^n \times \mathbb{P}^{n-r}$.

Solution to Exercise ??: The variety Z is just the universal hyperplane over the subvariety $\mathbb{P}^{n-r}$ of $\mathbb{G}(r,n)$; calling η the hyperplane class in $\mathbb{P}^n$, and σ the hyperplane class on $\mathbb{P}^{n-r}$; the class of Z will be of the form

$$[Z] = c_0\eta^{n-r} + c_1\eta^{n-r-1}\sigma + \ldots + c_{c-r}\sigma^{n-r},$$

and it is easy to see intersecting with a dual basis for $A_{n-k}(\mathbb{P}^n \times \mathbb{P}^{n-r})$ that all coefficients c_i are equal to 1 (trasversality will come from Kleiman's theorem). □

Exercise 1.203. **??** Let F and G be two general polynomials of degree 3 in $\mathbb{P}^2$, and let $\{C_t\}_{\mathbb{P}^1}$ be the associated pencil of curves; let $p_1, p_2, \ldots, p_9$ be the basepoints of these pencil. Show that for very general $t \in \mathbb{P}^1$ (that is, for all but countably many t), the line bundle $\mathcal{O}_{C_t}(p_1 - p_2)$ is not torsion in $\mathrm{Pic}(C_t) = A^1(C_t)$.

Solution to Exercise ??: Consider the total space $\pi : S \to \mathbb{P}^1$ of the pencil, that is the blow-up of $\mathbb{P}^2$ at the 9 points $p_1, \ldots, p_9$, and let let $E_1, \ldots, E_9$ be the exceptional divisors. For a given n, consider the line bundle $\mathcal{L}_n = \mathcal{O}_S(nE_1 - nE_2)$; if $p_1 - p_2$ is torsion of order n on every curve of the pencil, then the line bundle $\mathcal{L}_n$ is isomorphic to $\mathcal{O}_S$ on every fiber; using Corollary **??** (b), this means that $\mathcal{L}_n = \pi^*\mathcal{O}_{\mathbb{P}^1}(d)$; but on $Pic(S)$, this gives the equality

$$nE_1 - nE_2 = d(3H - E_1 - \ldots - E_9)$$

because the pullback of one point of $\mathbb{P}^1$ is a fiber, that means the proper transform of a cubic through the 9 base point. This cannot be true for any d and n, because H and all the E_i are linearly independent; hence, for every n the the divisor $p_1 - p_2$ is torsion of order n for finitely many curves; hence, outside of a countable set of fibers $p_1 - p_2$ is not torsion. □

Exercise 1.204. ?? Now let S be the blow-up of the plane at the points $p_1, \ldots, p_9$—that is, the graph of the rational map $\mathbb{P}^2 \to \mathbb{P}^1$ given by (F, G)—and let $E_1, \ldots, E_9$ be the exceptional divisors. Show that there is a biregular automorphism $\varphi : S \to S$ that commutes with the projection $S \to \mathbb{P}^1$ and carries E_1 to E_2.

Solution to Exercise ??: On the surface S, consider the two linear systems $3H - E_1 - \ldots - E_9$ (the one of the fibers for the map to $\mathbb{P}^1$) and $H - E_1$, of proper transforms of lines through p_1. Together, they give a map $S \to \mathbb{P}^1 \times \mathbb{P}^1$; this is a branched double cover: in fact, if we fix a line through p_1 and one of the curves of the pencil, there are going to be two points in the intersection, besides p_1. So, we can consider the involution φ_1 of S exchanging the two branches (hence, keeping curves of the pencil fixed). Consider also the involution φ_2, obtained in the same way using the system $H - E_2$ and the point p_2, and let us call φ the composition $\varphi_2 \circ \varphi_1$. Let us prove that this is the involution we want: if $r \in S$ (suppose it lies on a smooth curve of the pencil), we have $r + p_1 + \varphi_1(r) = 0$ by the group law on the curve, and $\varphi_1(r) + p_2 + \varphi(r) = 0$. Collecting everything, we get

$$\varphi(r) = r + p_2 - p_1$$

that holds on a dense subset, hence everywhere; this proves it carries E_1 to E_2. □

Exercise 1.205. ?? Using the result of Exercise **??**, show that the automorphism φ of Exercise **??** has infinite order, and deduce that the surface S contains infinitely many irreducible curves of negative self-intersection.

Solution to Exercise ??: Let us consider the curve E_1, and the images $\varphi^n(E_1)$ by the map in the previous exercise; as much as E_1, all curves $\varphi^n(E_1)$ will be curves with self intersection -1, and they will intersect every fiber in one reduced point (because φ commutes with the projection onto $\mathbb{P}^1$). Then, φ is an automorphism, so if $\varphi^n(E_1) = \varphi^m(E_1)$ for $n > m$, then $\varphi^{n-m}(E_1) = E_1$; but we cannot have $\varphi^{n-m}(E_1) = E_1$, because of Exercise **??**, so in the end all curves $\varphi^n(E_1)$ are distinct, and this proves the claim. □

Exercise 1.206. ?? An amusing enumerative problem: in the circumstances of the preceding exercises, for how many $t \in \mathbb{P}^1$ will it be the case that $\mathcal{O}_{C_t}(p_1 - p_2)$ is torsion of order 2—that is, that $\mathcal{O}_{C_t}(2p_1) \cong \mathcal{O}_{C_t}(2p_2)$?

Solution to Exercise ??: Consider the intersection $E_1 \cap \varphi^2(E_1)$; this will correspond to fibers where $p_1 = p_1 + 2(p_2 - p_1)$, that means where $2p_1 = 2p_2$. So, we have to understand what divisor is $\varphi^2(E_1) = \varphi(E_2)$, and in order to do this we will understand how φ acts on the Picard group of S. Remember that $\varphi = \varphi_1 \circ \varphi_2$, so let us start with φ_1. Remember that φ_1 exchanges points on lines through p_1 that lie on the same cubic curve of the pencil. On the exceptional divisors E_i with $i \neq 1$, the image will be the line

$\varphi_1(E_i) = H - E_1 - E_i$. About the exceptional curve E_1, its image will be of the kind

$$\varphi_1(E_1) = aH - bE_1 - \sum_{i_2}^{9} E_i$$

and imposing that the intersection with the image of E_i being zero, and self intersection to be -1, we get $\varphi_1(E_1) = 4H - 3E_1 - \sum E_i$. Doing the same for H, we get $\varphi_1(H) = 5H - 4E_1 - \sum E_i$. Formulas for φ_2 will be the same with E_1 and E_2 exchanged. So, we get

$$\begin{aligned}\varphi^2(E_1) &= \varphi_1\varphi_2\varphi_1\varphi_2(E_1) = \varphi_1\varphi_2\varphi_1(H - E_1 - E_2) = \\ &= \varphi_1\varphi_2(E_2) = \varphi_1(4H - E_1 - 3E_2 - \sum_{i=3}^{9} E_i) = \\ &= 6H - 3E_1 - \sum_{i=3}^{9} E_i\end{aligned}$$

whose intersection with E_1 gives 3, so that we have three fibers where the difference is torsion of order 2. In this way it is also possible to find the number of points of torsion of any order. □

Exercise 1.207. **??** Let C be a smooth curve of genus $g \geq 2$ over a field of characteristic $p > 0$; let $\varphi : C \to C$ be the Frobenius morphism. If $\Gamma_n \subset C \times C$ is the graph of φ^n and $\gamma_n = [\Gamma_n] \in A^1(C \times C)$ its class, show that the self-intersection $\deg(\gamma_n^2) \to -\infty$ as $n \to \infty$.

Solution to Exercise ??: Remember that the Frobenius morphism φ is bijective on closed points, but whose scheme theoretic fibers have degree p. To find the self-intersection of the graph of φ^n, we can use adjunction, that says

$$\deg(K_{\Gamma_n}) = \deg([\Gamma_n] \cdot (K_{C\times C} + [\Gamma_n]))$$

$$\deg([\Gamma_n]^2) = 2g - 2 - \deg([\Gamma_n] \cdot K_{C\times C}) = 2g - 2 - \deg([\Gamma_n] \cdot (K_{C\times 0} + K_{0\times C}))$$

where we used that, of course, Γ_n is a curve of genus g (because it is isomorphic to C itself). To find the remaining part of the formula, the divisor $K_{C\times 0}$ is $2g - 2$ fibers, so it will have intersection with Γ_n equal to $2g - 2$ (because Γ_n is a graph); the divisor $K_{0\times C}$ is $2g - 2$ fibers, so it will have intersection with Γ_n equal to $p^n(2g - 2)$ (because Γ_n is the graph of a morphism of degree p^n). We then get

$$\deg([\Gamma_n]^2) = -p^n(2g - 2)$$

that goes to $-\infty$ as n increases. □

Exercise 1.208. **??** Show that if E is a vector bundle of rank 2 and degree e on a smooth projective curve X, and L and M sub-line bundles of degrees a and b corresponding to sections of $\mathbb{P}E$ with classes σ and τ, then

$$\deg(\sigma\tau) = e - a - b$$

and

$$\deg(\sigma^2 + \tau^2) = 2e - 2a - 2b.$$

In particular, if L and M are distinct then $\deg\sigma^2 + \deg\tau^2 \geq 0$, with equality holding if and only if $E = L \oplus M$.

Solution to Exercise ??: Consider the vector bundles map $L \oplus M \xrightarrow{\iota} E$ given by the direct sum of the two immersions; a point of intersection of σ and τ will be a point of the curve X where the vector bundle map has rank 1 instead of two; that means, where the map $\wedge^2(L \oplus M) \xrightarrow{\wedge^2\iota} \wedge^2 E$, that is the same as the vanishing locus of a section of $\wedge^2(L \oplus M)^* \otimes \wedge^2 E$, whose first Chern class has in fact degree $e - a - b$. For the second equality, let us consider the sequence of morphisms

$$A^*(X) \xrightarrow{\pi^*} A^*(\mathbb{P}E) \xrightarrow{\pi_*} A^*(X).$$

This is an exact sequence, because of Theorem **??** and Exercise **??**; let us now consider it only in one degree, namely

$$A^1(X) \xrightarrow{\pi^*} A^1(\mathbb{P}E) \xrightarrow{\pi_*} A^0(X).$$

Consider the class $\sigma - \tau \in A^1(\mathbb{P}E)$: the general intersection with one fiber of $\mathbb{P}E$ is zero, so $\pi_*(\sigma - \tau) = 0$; but this means that $\sigma - \tau = \pi^*\alpha$ for $\alpha \in A^1(X)$, and hence that $(\sigma - \tau)^2 = 0$. Using what we found out about $\sigma\tau$, we get the desired

$$\deg(\sigma^2 + \tau^2) = 2e - 2a - 2b.$$

Now, if σ and τ are distinct, their intersection will be zero-dimensional, hence the intersection number must be positive; this proves $e - a - b \geq 0$, and hence $\deg\sigma^2 + \deg\tau^2 \geq 0$. In case of equality, σ and τ cannot intersect, and hence the map $L \oplus M \xrightarrow{\iota} E$ is an isomorphism of vector bundles. □

Exercise 1.209. **??** Using the analysis of Example **??** as a template, show that for $d > 1$ the universal hypersurface

$$\Phi_{d,n} = \{(X, p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in X\} \to \mathbb{P}^N$$

admits no rational section.

Solution to Exercise ??: Looking at the projection $\pi_2 : \Phi \to \mathbb{P}^n$, we can see that Φ

is a $\mathbb{P}^{N-1}$ bundle over $\mathbb{P}^n$. Calling η the hyperplane class of $\mathbb{P}^n$, and ζ the hyperplane section of $\mathbb{P}^N$, we get

$$A^*(\Phi) = \mathbb{Z}[\eta, \zeta]/(\eta^{n+1}, \zeta^N - p(\eta, \zeta)).$$

In particular, this means that

$$A^{n-1}(\Phi) = < \zeta^{n-1}, \eta\zeta^{n-2}, \ldots, \eta^{n-1} >_{\mathbb{Z}} .$$

Notice that if we intersect all these generators with the class of complementary dimension ζ^N, because of $\zeta^{N+1} = 0$ we get zero for all $\eta^{n-1-i}\zeta^i$ with $i > 0$, and intersecting with η^{n-1} we are asking, in a single hypersurface of degree d, how many points lie on a line: the answer is then d. Basically, we proved that *all codimension $n-1$ classes in Φ have intersection with ζ^N that is a multiple of d*. Now, suppose we have a rational section of the projection $\pi_1 : \Phi \to \mathbb{P}^N$; taking the closure of the image in Φ, this would give a codimension $n-1$ subvariety of Φ, having intersection one with the general fiber ζ^N, and this is not possible. □

Exercise 1.210. **??** Consider the flag variety of pairs consisting of a point $p \in \mathbb{P}^4$ and a 2-plane $\Lambda \subset \mathbb{P}^4$ containing p; that is,

$$\mathbb{F} = \{(p, L) : p \in \Lambda \subset \mathbb{P}^4\} \subset \mathbb{P}^4 \times \mathbb{G}(2, 4).$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^2$-bundle over $\mathbb{G}(2, 4)$, or as a $\mathbb{G}(1, 3)$-bundle over $\mathbb{P}^4$. Calculate the Chow ring $A(\mathbb{F})$ via each map, and show that the two descriptions agree.

Solution to Exercise ??: Over $\mathbb{G}(2, 4)$, the variety $\mathbb{F}$ may be viewed as $\mathbb{P}\mathcal{S}$. Its Chow ring is then

$$A^*(\mathbb{F}) = A * (\mathbb{G}(2, 4))[\zeta_0]/(\zeta_0^3 - \sigma_1\zeta_0^2 + \sigma_{1,1}\zeta - \sigma_{1,1,1}).$$

Over $\mathbb{P}^4$, the variety $\mathbb{F}$ can be seen as $G(2, Q)$; using the description as in Theorem **??**, we get

$$A^*(\mathbb{F}) = A^*(\mathbb{P}^4)[\zeta_1, \zeta_2]/(\eta^3 - \eta^2\zeta_1 + \eta(\zeta_1^2 - \zeta_2) + (-\zeta_1^3 + 2\zeta_1\zeta_2), \zeta_1^4 - 3\zeta_1^2\zeta_2 + \zeta_2^2).$$

And they are isomorphic by a ring morphism sending ζ_0 to η (they both represent the class of flags where the point belongs to an hyperplane) and σ_i to ζ_i (they both represent the class of flags where the plane intersect a $\mathbb{P}^{2-i}$). □

Exercise 1.211. **??** Show that the analog of Lemma **??** is false if we allow the V_i to have codimension > 1: in other words, $V_i \subset E_{p_i}$ is a general linear subspace of codimension m_i, then the corresponding subspace $W \subset H^0(E)$ need not have dimension $\max\{0, h^0(E) - \sum m_i\}$. (Hint: Consider a bundle whose sections all lie in a proper subbundle.)

Solution to Exercise ??: On the projective line $\mathbb{P}^1$, consider the vector bundle $E = \mathcal{O}(2) \oplus \mathcal{O}$, and let us impose the condition to lie in a codimension 2 subspace of the fiber (that is, to vanish) at two point p and q; on the 4 dimensional space of sections

$$H^0(E) \cong H^0(\mathcal{O}(2)) \oplus H^0(\mathcal{O})$$

this imposes three conditions: in fact, the first coordinate has to be zero, and the second has to be a polynomial vanishing at p and q (there is a one dimensional vector space of such sections). So in this case W has not the expected dimension (that in this case is 0). □

Exercise 1.212. **??** Calculate the remaining five intersection numbers in the table of intersection numbers in Section **??**.

Solution to Exercise ??: The class ω is the class of all conics contained in one of the planes of a net (that means, all conics that are coplanar with a given fixed point); the class γ is of a pencil of conics contained in a fixed plane, so it will not intersect ω; in the class φ, the plane varies in a pencil, so we are going to get one point of intersection. The class ζ on a general fiber of the projective bundle is going to be an hyperplane, and the class γ is a pencil in a general fiber: the intersection is clearly one again. For the class δ of conics meeting a line, the intersection with γ will be the only conic through the point of intersection of the line and the plane containing all conics of γ; the intersection with φ will be the two conics through the points of intersection of the line and the quadric we are taking hyperplane sections of. To prove transversality, we need to explicitly find tangent spaces; remember that the tangent space to $\mathcal{H}$ at a conic C is the vector space

$$H^0(\mathcal{N}_{C/\mathbb{P}^3}) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(1)|_C \oplus \mathcal{O}_{\mathbb{P}^3}(2)|_C) \cong H^0(\mathcal{O}_C(2)) \oplus H^0(\mathcal{O}_C(4)).$$

For δ, in the end, everything is carried on in Proposition **??**, and proof of what follows will be of the same kind. For γ, the tangent space is the vector space $0 \oplus V$ where V is the one dimensional subspace of polynomials vanishing at the four base points of the pencil. For φ, the tangent space is $W \oplus 0$, where W is the one dimensional subspace of polynomials vanishing at the two base points of the pencil (the intersection of the quadric and the line that is the base locus of the pencil of planes). For ω, the tangent space is $W' \oplus H^0(\mathcal{O}_C(4))$, where W' is the pencil of line sections of the conic passing through the point in the plane that is the base locus of the net of planes of ω. For ζ, we really do not use it as a variety, but only as Chern class of a line bundle (that works as normal bundle), so we do not need it. □

Exercise 1.213. **??** To find the class $\delta = [D_L] \in A^1(\mathcal{H})$ of the cycle of conics meeting a line directly, restrict to the open subset $U \subset \mathcal{H}$ of pairs $(H, \xi) \in \mathcal{H}$ such that H does not contain L (since the complement of this open subset of $\mathcal{H}$ has codimension 2, any relation among divisor classes that holds in U will hold in $\mathcal{H}$). Show that we have a map

$\alpha : U \to L$ sending a pair (H, ξ) to the point $p = H \cap L$, and that in U the divisor D_L is the zero locus of the map of line bundles

$$T \to \alpha^* \mathcal{O}_L(2)$$

sending a quadric $Q \in \xi$ to $Q(p)$.

Solution to Exercise ??: The definition we give for the line bundle map does not necessarily imply that the target line bundle has to be $\alpha^* \mathcal{O}_L(2)$, but only that is constant along fiber of α, that means, of the kind $\alpha^* \mathcal{O}_L(d)$; to verify that the shift is actually 2, let us consider a pencil of conics of type φ, that means, the intersection of a quadric Q and a pencil of planes (noone of them containing L); note that φ maps to L by α isomorphically; on φ, the bundle T is trivial (because we have a nonvanishing section given by the polynomial defining Q), hence the line bundle map can be seen as just a section of $\alpha^* \mathcal{O}_L(d) \cong \mathcal{O}_\varphi(d)$; this map has two zeroes, at the two planes containing the two intersection point of L and Q: this proves that $d = 2$. It is immediate now to prove that the vanishing locus of this map is indeed D_L. This gives also a different way to find the class $[D_L]$; noticing that the first Chern class of $\alpha^* \mathcal{O}_L(1)$ is exacly the class ω, we immediately get $[D_L] = 2\omega + \zeta$. □

Exercise 1.214. ?? Let $\Delta \subset \mathcal{H}$ be the locus of singular conics.

(a) Show that Δ is an irreducible divisor in $\mathcal{H}$.
(b) Express the class $\delta \in A^1(\mathcal{H})$ as a linear combination of ω and ζ.
(c) Use this to calculate the number of singular conics meeting each of 7 general lines in $\mathbb{P}^3$; and
(d) Verify your answer to the last part by calculating this number directly.

Solution to Exercise ??: In every plane, the set of singular conics is an irreducible variety (for instance, it can be seen as the image of $\mathbb{P}^{2*} \times \mathbb{P}^{2*}$ in $\mathbb{P}^5$); so, every fiber of the fibration $\Delta \to \mathbb{P}^{3*}$ is an irreducible divisor, hence Δ is too. To find its class, we need to intersect with pencils γ and ω; it is easy to see that intersections are respectively 3 (the numbber of singular plane conics in a pencil) and 2 (the number of singular hyperplane sections of a quadric surface): this gives us

$$[\Delta] = 2\omega + 3\zeta.$$

To find the number of singular conics meeting 7 lines, we need to find the degree of the product

$$(2\omega + 3\zeta)(2\omega + \zeta)^7 = 3\zeta^8 + 44\omega\zeta^7 + 280\omega^2\zeta^6 + 448\omega^3\zeta^5$$

That gives us

$$3 \cdot (-4) + 44 \cdot 6 + 280 \cdot (-4) + 448 = 140.$$

We cannot use transversality, because this would need studying $H^0(\mathcal{N}_{C/\mathbb{P}^3})$ where C is a nodal space curve, that would require more technical results. Without transversality, we get that 140 is the number of intersection points counted with multiplicity (it is easy to prove that the intersection actually happens in a zero dimensional variety); hence, the number of such conics is less than or equal to 140. Considering the geometry of lines in $\mathbb{P}^3$, a singular conic will be the union of two intersecting lines; there is no line through 5 of the lines of the 7 we are considering, so in our conic one component L_1 has to meet 4 of the lines, and the other L_2 has to meet the other three; now, to choose L_1 we need to pick 4 lines out of the seven, and then we have two choices, because of Schubert calculus; for L_2, we need it to intersect the other 3 lines, and L_2, giving another problem of Schubert calculus that still has 2 solutions. We then get

$$\binom{7}{4} \cdot 2 \cdot 2 = 140$$

different conics, that together with the previous part proves that the number is exactly 140. □

Exercise 1.215. **??** Let $p \in \mathbb{P}^3$ be a point, and $F_p \subset \mathcal{H}$ the locus of conics containing the point p. Show that F_p is six-dimensional, and find its class in $A^2(\mathcal{H})$

Solution to Exercise ??: Let us consider the projection $F_p \to \mathbb{P}^{3*}$; over planes not containing p, the fiber is empty; over planes containing p, the fiber is an hyperplane. This proves that F_p is irreducible and six-dimensional. To find its class, let us restric ourselves to the subvariety Z of conics contained in a plane that contains p; on this, we have a map as in Exercise **??**

$$T \to \mathcal{O}$$

still obtained by $Q \mapsto Q(p)$, where now the target is just the trivial bundle (it is the pullback of the structure sheaf of the point p), and F_p is the vanishing locus of this map, hence its class on Z is the pullback from $\mathcal{H}$ of ζ; now we can use the push pull formula for the embedding $i : Z \hookrightarrow \mathcal{H}$: we have

$$[F_p] = i_*([Z] \cdot i^*\zeta) = i_*([Z]) \cdot \zeta = \omega\zeta.$$

□

Exercise 1.216. **??** Use the result of the preceding exercise to find the number of conics passing through a point p and meeting each of 6 general lines in $\mathbb{P}^3$, the number of conics passing through two points p, q and meeting each of 4 general lines in $\mathbb{P}^3$, and the number of conics passing through three points p, q, r and meeting each of 2 general lines in $\mathbb{P}^3$. Verify your answers to the last two parts by direct examination.

Solution to Exercise ??: We have

$$\deg(\omega\zeta(2\omega+\zeta)^6) = \deg(\omega\zeta^7 + 12\omega^2\zeta^6 + 60\omega^3\zeta^5) = 18$$

$$\deg(\omega^2\zeta^2(2\omega+\zeta)^4) = \deg(\omega^2\zeta^6 + 8\omega^3\zeta^5) = 4$$

$$\deg(\omega^3\zeta^3(2\omega+\zeta)^2) = \deg(\omega^3\zeta^5) = 1.$$

To prove that these are actually transverse, it is possible to prove (as in Proposition **??**) that the tangent space to F_p is the space of normal vector fields that vanish at p; also, using the fact that the normal bundle $\mathcal{O}_C(2) \oplus \mathcal{O}_C(4)$ it is possible to prove that intersection is still transverse (so, everything imposes the right number of conditions even though we are not anymore in the hypothesys of Lemma **??**). For the last one, there is a unique plane H through the three points p, q, r, and a unique conic on H through p, q, r and the two points of intersection between H and the two lines. To prove the middle one, consider the conics through p, q and three of the lines, l_1, L_2, L_3; this is a one parameter family of lines, one for each of the planes containing p and q; together they form a surface S in $\mathbb{P}^3$, and the result will follow as the intersection with the fourth line L_4, that is, the degree of S, that is 4. □

Exercise 1.217. ?? Find the class in $A^3(\mathcal{H})$ of the locus of double lines (note that this is five-dimensional, not four!)

Solution to Exercise ??: Let us choose a different basis for the Chow ring of $\mathcal{H}$. Instead of ω, ζ, we will use ω, δ; the class of this locus will be

$$[Y] = c_0\omega^3 + c_1\omega^2\delta + c_2\omega^1\delta^2 + c_3\delta^3.$$

We want to intersect with classes δ^5, $\omega\delta^4$, $\omega^2\delta^3$ and $\omega^3\delta^2$; first, we need to find intersection numbers δ^8, $\omega\delta^7$, $\omega^2\delta^6$ and $\omega^3\delta^5$; these are respectively 92,34,8 and 1. The intersection of Y with δ^5, that is zero because no (double) line meets five lines, gives rise to

$$0 = c_0 + 8c_1 + 34c_2 + 92c_3.$$

The intersection with $\omega\delta^4$, that is 2 because two lines meet 4 lines (and then we choose the plane in the only way it is in the net given by ω), we get

$$2 = c_1 + 8c_2 + 34c_3.$$

To find the intersection with $\omega^2\delta^3$, notice that there is a one parameter family of lines meeting three general lines, the ones of one ruling of the unique quadric containing them. Then, we want it to be contained in a pencil of planes, that means, to meet a fixed line, and this happens for two of the lines of this ruling; we then get

$$2 = c_2 + 8c_3.$$

The intersection with $\omega^3\delta^2$ is obviously 1 (we choose the plane, and we have the two points to join to get a line); we get then

$$1 = c_3.$$

Building everything back again, we get

$$[Z] = \delta^3 - 6\omega^1\delta^2 + 16\omega^2\delta - 15\omega^3 = \zeta^3 + 4\omega^2\zeta + \omega^3.$$

We will forget transversality issues in this case. □

Exercise 1.218. **??** Suppose that $X \subset \mathbb{P}^n$ is a subscheme of pure dimension l, and $\mathcal{H}$ a component of the Hilbert scheme parametrizing subschemes of $\mathbb{P}^n$ of pure dimension $k < n - l$ in $\mathbb{P}^n$; let $[Y] \in \mathcal{H}$ be a smooth point corresponding to a subscheme $Y \subset \mathbb{P}^n$ such that $Y \cap X = \{p\}$ is a single reduced point, and suppose moreover that p is a smooth point of both X and Y. Finally, let $\Sigma_X \subset \mathcal{H}$ be the locus of subschemes meeting X.

Use the technique of Proposition **??** to show that $\Sigma_X \subset \mathcal{H}$ is smooth at $[Y]$, of the expected codimension $n - k - l$, with tangent space

$$T_{[Y]}\Sigma_X = \left\{\sigma \in H^0(N_{Y/\mathbb{P}^n}) : \sigma(p) \in \frac{T_pX + T_pY}{T_pY}\right\}.$$

Solution to Exercise ??: This exercise is left to the reader. □

The next few problems deal with an example of a phenomenon encountered in the preceding chapter: the possibility that the cycles in our parameter space corresponding to the conditions imposed in fact do not meet transversely, or even properly.

Exercise 1.219. **??** Let $H \subset \mathbb{P}^3$ be a plane, and let $\mathcal{E}_H \subset \mathcal{H}$ be the closure of the locus of smooth conics $C \subset \mathbb{P}^3$ tangent to H. Show that this is a divisor, and find its fundamental class $\beta \in A^1(\mathcal{H})$.

Solution to Exercise ??: Using an argument similar to the one in Proposition **??**, if C is a curve tangent at p to H, then

$$T_{[Y]}\mathcal{E}_X = \left\{\sigma \in H^0(N_{C/\mathbb{P}^3}) : \sigma(p) \in \frac{T_pH}{T_pC}\right\}.$$

In order to see this, we need to consider the incidence correspondence Ψ_H of couples $(\tilde{p}, C)$ of a degree 2 subscheme of H supported in a point (that means, a point with a direction sticking out) and a curve C containing this degree 2 subscheme; note that is exactly the condition for C to be tangent to H. Considering normal bundles and applying Lemma **??** (note that $H^0(\mathcal{N}_{\tilde{p}/H})$ is different from T_pH!), we can prove the claim.

Using this, we can prove transversality with general cycles γ and φ, and just count the number of intersection points; with γ, we need the number of conics in a pencil tangent to a line, that is 2; with φ, we need the number of hyperplane sections of a quadric in a pencil that are tangent to a plane; or, the number of lines in a pencil in a plane that are tangent to a conics: the answer is again two. We then find

$$[\mathcal{E}_H] = 2\omega + 2\zeta$$

that is the class we were looking for. Note that the entire variety of double lines is contained in □

Exercise 1.220. ?? Find the number of smooth conics in $\mathbb{P}^3$ meeting each of 7 general lines $L_1, \dots, L_7 \subset \mathbb{P}^3$ and tangent to a general plane $H \subset \mathbb{P}^3$. More generally, find the number of smooth conics in $\mathbb{P}^3$ meeting each of $8-k$ general lines $L_1, \dots, L_{8-k} \subset \mathbb{P}^3$ and tangent to a k general planes $H_1, \dots, H_k \subset \mathbb{P}^4$, for $k = 1, 2$ and 3.

Solution to Exercise ??: For $k \leq 3$, then it is easy to prove that there is no double line in the intersection (because no line meets 5 general lines), and that there is no singular conic either, because for the union of two incident lines the only option to lie in $\mathcal{E}_H$ is either to have one component entirely contained in H, or to have the node lying on H; neither of this is possible if $k \leq 3$; so, in the smooth conics in the intersection the cycles are going to intersect transversely, so we just need to find the degree of the intersection of cycles, that is

$$\deg((2\omega + 2\zeta)(2\omega + \zeta)^7) = 116$$

$$\deg((2\omega + 2\zeta)^2(2\omega + \zeta)^6) = 128$$

$$\deg((2\omega + 2\zeta)^3(2\omega + \zeta)^5) = 104.$$

□

Exercise 1.221. ?? Why don't the methods developed here work to calculate the number of smooth conics in $\mathbb{P}^3$ meeting each of $8-k$ general lines $L_1, \dots, L_{8-k} \subset \mathbb{P}^3$ and tangent to a k general planes $H_1, \dots, H_k \subset \mathbb{P}^4$, for $k \geq 4$? What can you do to find these numbers? (In fact, we have seen how to deal with this in Chapter **??**)

Solution to Exercise ??: For $k \geq 5$, a positive family of double lines appears appears in the intersection, so intersection will not be transverse anymore. For $k = 4$, we still have two double lines in the intersection, and they will not be reduced points of the intersection. One solution, as in the case of plane conics, is to blow up the locus of double conics, or to consider the moduli space of stable maps $\overline{M}_{0,0}(\mathbb{P}^3, 2)$; in this case, they will give two different compactifications of the space of smooth space conics. □

Next, some problems involving conics in $\mathbb{P}^4$:

Exercise 1.222. **??** Now let $\mathcal{K}$ be the space of conics in $\mathbb{P}^4$ (again, defined to be complete intersections of two hyperplanes and a quadric). Use the description of $\mathcal{K}$ as a $\mathbb{P}^5$-bundle over the Grassmannian $\mathbb{G}(2,4)$ to determine its Chow ring.

Solution to Exercise ??: As in the case of $\mathbb{P}^3$, $\mathcal{K}$ is the space $\mathbb{P}(\mathrm{Sym}^2\,\mathcal{S}^*)$; the total Chern class of $\mathrm{Sym}^2\,\mathcal{S}^*$ is

$$c(\mathrm{Sym}^2\,\mathcal{S}^*) = 1+4\sigma_1+10\sigma_2+5\sigma_{1,1}+15\sigma_{2,1}+21\sigma_{1,1,1}+10\sigma_{2,2}+30\sigma_{2,1,1}+20\sigma_{2,2,1}$$

hence we have

$$A^*(\mathbb{G}(2,4))[\zeta]/(\zeta^6+4\sigma_1\zeta^5+10\sigma_2\zeta^4+5\sigma_{1,1}\zeta^4+15\sigma_{2,1}\zeta^3+21\sigma_{1,1,1}\zeta^3+10\sigma_{2,2}\zeta^2+30\sigma_{2,1,1}\zeta^2+20\sigma_{2,2,1}$$

that describes completely the Chow ring of $\mathcal{K}$. □

Exercise 1.223. **??** In terms of your answer to the preceding problem, find the class of the locus D_Λ of conics meeting a 2-plane Λ, and of the locus $\mathcal{E}_L$ of conics meeting a line $L \subset \mathbb{P}^4$.

Solution to Exercise ??: Let us use a table such as the one in Section **??**. Consider the pencil γ of conics lying in a plane, and the pencil φ of plane sections of a general quadric surface in an hyperplane; the table turns out to be exactly the same.

	σ_1	ζ	$[D_\Lambda]$
γ	0	1	1
φ	1	0	2

All checks are easy, as in Exercise **??** (for $\zeta\varphi$ we can use the same argument as in Section **??**; the class is then, again, $[D_\Lambda] = 2\sigma_1 + \zeta$. □

Exercise 1.224. **??** Find the expected number of conics in $\mathbb{P}^4$ meeting each of 11 general 2-planes $\Lambda_1, \ldots, \Lambda_{11} \subset \mathbb{P}^4$.

Solution to Exercise ??: We need to find the degree $(2\sigma_1 + \zeta)^{11}$; with the aid of the *Schubert2* package of Macaulay2 (available at **?**]), we found this degree to be 6620. □

Exercise 1.225. **??** Prove that your answer to the preceding problem is in fact the actual number of conics by showing that for general 2-planes $\Lambda_1, \ldots, \Lambda_{11} \subset \mathbb{P}^4$ the corresponding cycles D_{Λ_i} intersect transversely.

Solution to Exercise ??: We leave this exercise to the reader, that can be solved proving that the in the intersection there are only smooth conics, then proving an equivalent of Proposition **??** for conics in $\mathbb{P}^4$ and invoking Lemma **??**. □

Finally, here's a challenge problem:

Exercise 1.226. **??** Let $\{S_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ be a general pencil of quartic surfaces (that is, take A and B general homogeneous quartic polynomials, and set $S_t = V(t_0 A + t_1 B) \subset \mathbb{P}^3$). How many of the surfaces S_t contain a conic?

Solution to Exercise ??: Consider the line bundle $\mathcal{O}_{\mathbb{P}^3}(4)$; we would like a vector bundle $\mathcal{E}$ such that for every $C \in \mathcal{H}$, the fiber is the 9 dimensional vector space $H^0(\mathcal{O}_{\mathbb{P}^3}(4)|_C)$; in this way, a quartic polynomial would be a section of $\mathcal{E}$, and a section vanishing at C means a quartic surface containing C. So, we expect a pencil of sections to have a finite number of zeroes, and this number being evaluated by $c_8(\mathcal{E})$. We need then to describe $\mathcal{E}$, and we will obtain it as a quotient of $H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \otimes \mathcal{O}_{\mathcal{H}} \cong \mathcal{O}_{\mathcal{H}}^{35}$, a trivial bundle, and a vector bundle whose fiber over a curve C is the space of all polynomials of degree 4 vanishing on C. Let us call L be the polynomial of degree 1 on $\mathbb{P}^3$ vanishing on C, and Q the polynomial of degree 2 on the plane H defined by L that vanishes on C (note that there is not a single quadric polynomial in $\mathbb{P}^3$ vanishing on C, it is unique only up to a multiple of L). Among polynomials of degree 4 on $\mathbb{P}^3$ vanishing on C, there are all multiples of L by a polynomial of degree 3; remembering that the multiples of the polynomial L compose the fiber of the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^{3*}}(-1)$ that has Chern class $-\omega$, and that polynomials of degree 3 are just a trivial bundle of rank 20, taking the tensor product we get the vector bundle $\mathcal{F}_1 = \mathcal{O}_{\mathcal{H}}(-\omega)^{20}$. The fiber of the vector bundle $\mathcal{O}_{\mathcal{H}}^{35}/\mathcal{O}_{\mathcal{H}}(\omega)^{20}$ over the point C is the vector space of polynomials of degree 3 on the plane C lies in. We need then to take out the products of Q (coming from the line bundle $\mathcal{O}_{\mathcal{H}}(-\zeta)$) with polynomials of degree 2 on this plane; polynomials of degree 2 on this plane will be polynomials of degree 2 on $\mathbb{P}^3$ (hence, a 10-dimensional trivial vector bundle) quotiented out by polynomials of degree 2 that are multiple of L, that means (hence, a 4-dimensional trivial vector bundle tensor the line bundle of degree $-\omega$). We hence get the vector bundle

$$\mathcal{F}_2 = (\mathcal{O}_{\mathcal{H}}^{10}/\mathcal{O}_{\mathcal{H}}(-\omega)^4) \otimes \mathcal{O}_{\mathcal{H}}(-\zeta) \cong \mathcal{O}_{\mathcal{H}}(-\zeta)^{10}/\mathcal{O}_{\mathcal{H}}(-\omega-\zeta)^4.$$

Collecting everything, we have

$$\mathcal{E} = (\mathcal{O}_{\mathcal{H}}^{35}/\mathcal{F}_1)/\mathcal{F}_2$$

whose total Chern class is given by

$$\begin{aligned} c(\mathcal{E}) &= \frac{1}{c(\mathcal{F}_1)c(\mathcal{F}_2)} = \\ &= \frac{(1-\omega-\zeta)^4}{(1-\omega)^2 0(1-\zeta)^1 0} \end{aligned}$$

and after the calculation, we get

$$c_8(\mathcal{E}) = 89892\omega^3\zeta^5 + 33396\omega^2\zeta^6 + 8976\omega\zeta^7 + 1287\zeta^8$$

that has degree 5016. Finally, notice that every quartic surface containing one conic also contain another, because if an hyperplane section contains a conic, then the residual curve is another conic, so the expected number of such quartics is 2508. Transversality follows from Lemma **??** (b), because $\mathcal{E}$ is generated by global sections; it is also possible to prove with an incidence correspondence that no conic appearing in this pencil is singular (hence, those elements of the pencil have nothing to do with the elements containing a line). □

1.10 Chapter 10

Exercise 1.227. **??** Use the result of Exercise **??** (describing the class of the universal k-plane in $\mathbb{P}^r \times \mathbb{G}(k,r)$) to give an alternative proof of Proposition **??**.

Solution to Exercise ??: From Exercise **??**, the class of the universal k-plane $\mathbb{F}$ in $\mathbb{P}^r \times \mathbb{G}(k,r)$ is

$$[\mathbb{F}] = \sum_{j=0}^{r-k} \zeta^{r-k-j} \sigma_j.$$

Take now $B \subset \mathbb{G}(k,r)$ of dimension m; the variety swept out by planes in B can be obtained taking the inverse image $\mathcal{B}$ of B by π_2 in $\mathbb{F}$, and then the image X of $\mathcal{B}$ by π_2 in $\mathbb{P}^r$. To find the class, we have

$$[\mathcal{B}] = [\mathbb{F}] \cdot \pi_2^*[B] = \sum_{j=0}^{r-k} \zeta^{r-k-j} (\sigma_j \cdot [B])$$

$$[X] = \pi_{1*}[\mathcal{B}]/d$$

where d is the degree of the (generically finite) map $\mathcal{B} \to X$. Now classes with nonzero pushforward are those of the kind $\sigma_\lambda \zeta^i$ where σ_λ is a zero dimensional class. The only summand of $[\mathcal{B}]$ that will end up with a zero dimensional Schubert cycle is $\zeta^{r-k-m}(\sigma_m \cdot [B])$, whose pushforward will be

$$[X] = (\deg(\sigma_m \cdot [B])/d)\zeta^{r-k-m}$$

so we get the degree of X is $\deg(\sigma_m \cdot [B])/d$; but now we know by Proposition **??** that on $\mathbb{G}(k,r)$ we have

$$s_m(\mathcal{S}) = c_m(\mathcal{Q}) = \sigma_m,$$

and we also have so that this degree is also equal to the degree of the m-th Segre class of the bundle $\mathcal{S}$ when restricted to B, as we needed to prove. □

Exercise 1.228. **??** Let $X \subset \mathbb{P}^r$ be a variety, and $\Sigma_m(X) \subset \mathbb{G}(m-1, r)$ the image of the secant plane map $\tau : X^{(m)} \to \mathbb{G}(m-1, r)$. Show by example that not every $(m-1)$-plane Λ such that $\deg(\Lambda \cap X) \geq m$ lies in $\Sigma_m(X)$. (For example, try X a curve in $\mathbb{P}^5$ with a trisecant line, with $m = 3$.)

Solution to Exercise ??: As suggested, let X be a curve in $\mathbb{P}^5$ with a trisecant line L meeting X at p, q, r (we will say a couple of words in the end about how to obtain such a curve); suppose that the trisecant lines is isolated, meaning that the rational map τ is not defined, in a neighborhood of $p + q + r$, only at $p + q + r$; to have this condition, it is enough for instance to have the three tangent lines $\mathbb{T}_p X$, $\mathbb{T}_q X$ and $\mathbb{T}_r X$ to generate an hyperplane H. The map τ is not defined at $p + q + r$, because there is not an unique 2-plane containing the three points; there is in fact a three dimensional space of such planes, the Schubert cycle $\Sigma_{3,3}(L)$. Now, when we take the closure of the image of τ at $p + q + r$, the new stuff that will be added will lie inside this $\Sigma_{3,3}(L)$, because every plane will be limit of planes meeting X with multiplicity 3; but, we are taking the closure of a 3-dimensional variety, so the "stuff" we are adding can have dimension at most 2, so it cannot cover the entire $\Sigma_{3,3}(L)$ (in fact, it is easy to prove that we are adding the cycle $\Sigma_{3,3,1}(L, H)$); this proves than $\Sigma_3(X)$ does not contain all planes meeting X at a scheme of degree 3. To obtain this curve, we can for instance take the projection of a rational normal curve Y of degree 6 in $\mathbb{P}^6$, projecting from a point contained in a single 3-secant plane; it is possible to prove that Y has no 3-secant line, and that a general point in a general 2-secant plane lies only in one single such plane, hence that on the projection we have a single isolated 3-secant line. □

Exercise 1.229. **??** Prove Proposition **??** in the case of a nondegenerate space curve $C \subset \mathbb{P}^3$—that is, that the line joining two general points of C does not meet the curve a third time—without using the general position lemma (**??**).

Solution to Exercise ??: Consider two general points p and q of C, and consider the line L joining them; if L meets the curve C also at a third point r, then it is easy to prove that $\mathbb{T}_p C$ and $\mathbb{T}_q C$ intersect (otherwise in a neighborhood of L in $\mathbb{G}(1, 3)$ we would have secants to C that are not trisecant). Hence, all tangent lines to C intersect each other, and this means that either they lie in the same plane, or they pass through the same point (this is easy to check, and holds for any set of lines pairwise intersecting, not necessarily infinite). If the lines are all in the same plane, then the curve is too, hence is not nondegenerate; if all tangent lines are through the same point, then projecting away from that point we would have a map $C \to \mathbb{P}^2$ whose differential is zero at every point, hence is constant (that means that C is a line). Note that the last implication is not true anymore in positive characteristic XXXX CHECK BACK WITH JOE IF THE ENTIRE RESULT IS TRUE. □

In Exercises **??**-**??** we'll verify that the Veronese varieties listed in Theorem **??** are indeed defective.

Exercise 1.230. **??** Show that for $p, q \in \mathbb{P}^n$, the subspace $H^0(\mathcal{I}_p^2\mathcal{I}_q^2(2)) \subset H^0(\mathcal{O}_{\mathbb{P}^n}(2))$ of quadrics singular at p and q has codimension $2n+1$ (rather than the expected $2n+2$). Deduce that any two tangent planes to the quadratic Veronese variety $\nu_2(\mathbb{P}^n)$ meet, and thus that $\nu_2(\mathbb{P}^n)$ is 2-defective for any n.

Solution to Exercise ??: Note that the condition of being singular at a point p, for an hypersurface in $\mathbb{P}^n$, is a condition of codimension $n+1$; for a quadric hypersurface, being singular at two points is the same as being singular along the line joining them, so it is a condition of codimension $2n+1$ (we can see the condition "missing" from the fact that once the hypersurface is singular at one point, and it contains another point, it contains the entire line joining them).

Remember that in the Veronese embedding $\nu_2 : \mathbb{P}^n \to \mathbb{P}^N$, hyperplane sections cut on $\nu_2(\mathbb{P}^n)$ all quadric hypersurfaces; hyperplanes containing the tangent plane at $\nu_2(p)$ are those cutting out hypersurfaces singular at p. For two general points $p, q \in \mathbb{P}^r$, if $\mathbb{T}_p\nu_2(\mathbb{P}^n)$ and $\mathbb{T}_q\nu_2(\mathbb{P}^n)$ were independent, then they would impose $2(n+1)$ conditions on hyperplanes to contain both of them; from what we have said so far, they are not independent (hence, they meet), so using Terracini's lemma we can conclude that $\nu_2(\mathbb{P}^n)$ is 2-defective for any n. □

Exercise 1.231. **??** Show that for any five points $p_1, \dots, p_5 \in \mathbb{P}^2$, there exists a quartic curve double at all five; deduce that the tangent planes $\mathbb{T}_{p_i}S$ to the quartic Veronese surface $S = \nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ are dependent (equivalently, fail to span $\mathbb{P}^{14}$), and hence that S is 5-defective.

Solution to Exercise ??: As before, we need to prove that the five tangent planes $\mathbb{T}_{p_i}S$ are dependent; the condition of being singular at a point, for a plane quartic, is a codimension three conditions; to have the planes dependent, we need the condition to be tangent at 5 points to have codimension 15; in particular, as we saw, plane quadrics form a $\mathbb{P}^{14}$, and we should not have *any* plane quartic singular at 5 points; the point is, we have one: is the double of the unique conic through the points $p_1, \dots, p_5$; this proves that the five planes $\mathbb{T}_{p_i}S$ are not independent, and by Terracini's lemma S is 5-defective. □

Exercise 1.232. **??** Show that for any nine points $p_1, \dots, p_9 \in \mathbb{P}^3$, there exists a quartic surface double at all nine; deduce that the tangent planes $\mathbb{T}_{p_i}X$ to the quartic Veronese threefold $X = \nu_4(\mathbb{P}^3) \subset \mathbb{P}^{34}$ fail to span $\mathbb{P}^{34}$), and hence that X is 9-defective.

Solution to Exercise ??: Exactly as in the previous exercise, every tangent space imposes 4 conditions, so if the planes $\mathbb{T}_{p_i}X$ were independent, they would impose 36 conditions; they do not, in fact we can consider the double of the unique quadric through the 9 points, that gives an hyperplane containing the 9 planes; the claim follows again from Terracini's lemma. □

Exercise 1.233. **??** Finally, show that for any seven points $p_1, \dots, p_7 \in \mathbb{P}^4$, there exists a cubic threefold double at all seven; deduce that the tangent planes $\mathbb{T}_{p_i} X$ to the cubic Veronese fourfold $X = \nu_3(\mathbb{P}^4) \subset \mathbb{P}^{34}$ are dependent (equivalently, fail to span $\mathbb{P}^{34}$), and hence that X is 7-defective. (Hint: this problem is harder than the preceding three; you have to use the fact that through seven general points in $\mathbb{P}^4$ there passes a rational normal quartic curve.)

Solution to Exercise ??: Let us for now suppose that we know that through 7 general points $p_1, \dots, p_7 \in \mathbb{P}^4$ there is a twisted cubic C (in fact, there is only one), we will prove it later. Now, let us consider the secant variety $Sec_2(C)$; this is 3-dimensional, and has degree 3 (it can be proved projecting C away from a line, or just by Theorem **??**). $Sec_2(C)$ is singular along C, because at a point p of C the tangent cone is just the union of all lines joining p and other points of C; hence, it is singular at the 7 points $p_1, \dots, p_7$. Now, for an hypersurface in $\mathbb{P}^4$, being singular at a point p is a codimension 5 condition, so at 7 points it should be a 35 codimensional condition; we should not have a cubic hypersurface singular at 7 points, but in fact we have; for the same reason as in the previous exercises, then, X is 7-defective. To show that there is a rational normal curve through 7 general points, we will show the Steiner construction (that in fact works for rational normal curves through $2d-1$ points in $\mathbb{P}^d$): let us label the points p_0, p_1, p_∞ and q_1, q_2, q_3, q_4; consider now the four pencils of 3-planes $\{H_{i,t}\}$ for $i = 1, 2, 3, 4$ containing the points q_1, q_2, q_3, q_4 besides q_i; let us parametrize these pencils using a parameter t in such a way that $H_{i,0}$ contains the point p_0, $H_{i,1}$ contains the point p_1 and $H_{i,\infty}$ contains the point p_∞. Consider now the curve obtained as

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^4 \\ t &\mapsto \bigcap_{i=1}^{4} H_{i,t}. \end{aligned}$$

It is now an easy exercise (for instance, using Schubert calculus) to prove that this actually gives a rational normal curve, that of course will pass through the seven points. □

The following exercises can be solved using the following fact, the *completeness of the adjoint series* for plane curves: if C is a nodal curve of degree d in $\mathbb{P}^2$, and $\tilde{C}$ its normalization, then we obtain the entire canonical series $H^0(K_{\tilde{C}})$ pulling back polynomials of degree $d-3$ on $\mathbb{P}^2$ vanishing on the nodes of C.

Exercise 1.234. **??** Show that the twisted cubic curve is the unique nondegenerate curve $C \subset \mathbb{P}^3$ such that a general point $p \in \mathbb{P}^3$ lies on a unique secant line to C. (Note: this can be done without it, but it's easy if you apply the *Castelnuovo bound* on the genus of a curve in $\mathbb{P}^3$; see Chapter 3 of **?**] for a statement and proof.)

Solution to Exercise ??: Let C be a curve of degree d and genus g with this property, and let us project the curve away from a general point p of $\mathbb{P}^3$ onto $\mathbb{P}^3$; nodes of the image C' of C will correspond to secant lines to C that contain p, so by hypothesis the curve C' will have only one node. We are now going to use the completeness of the adjoint series, to prove the following claim: for a plane curve C' of degree d with δ nodes (let us denote by C its normalization), if $\delta \leq d-3$ then $h^0(\mathcal{O}_C(1)) = 3$; in particular, this means that the embedding of the curve C is normal and then it cannot come from a projection of a curve in $\mathbb{P}^3$; applied to our case, we have a plane curve with one node, so the only option for it to come from a projection is if $d \leq 3$, and being nondegenerate the only option is for it to be the twisted cubic. Let us now prove the claim; by Riemann-Roch, we have

$$h^0(\mathcal{O}_C(1)) = d - g + 1 + h^0(K \otimes \mathcal{O}_C(-1)).$$

But now, sections of $K \otimes \mathcal{O}_C(-1)$ are just polynomials of degree $d-3$ vanishing on the δ nodes of C' and the d points of intersection of C' with a line L. Suppose now that these $d+\delta$ points fail to impose independent conditions on polynomials of degree $d-3$ by a number η (that means, the condition of passing through all of them is a condition of codimension $d+\delta-\eta$); plugging in everything in the equation, and using that $g = \binom{d-1}{2} - \delta$, we get the simpler formula

$$h^0(\mathcal{O}_C(1)) = 1 + \eta$$

that sometimes is called *geometric Riemann-Roch formula*; let us finally prove that if $\delta \leq d-3$, then $\eta = 2$; for this, notice that a polynomial of degree $d-3$ vanishing on d points of a line L, is going to vanish on the entire line (that is a condition of codimension $d-2$, so have "failures" here); so, we ask for polynomials of degree $d-4$ vanishing on $\delta \leq d-3$ points: here, the conditions have to be independent, because for instance we can find polynomials of degree $d-4$ vanishing on all points but one (just taking union of random lines through all of them but one), so here we have no "failure", and $\eta = 2$, that proves the clais □

Exercise 1.235. ?? Show that the rational normal curve and the elliptic normal curve of degree $d+1$ are the only nondegenerate curves $C \subset \mathbb{P}^d$ with the property that every divisor of degree d on C spans a hyperplane.

Solution to Exercise ??: Let us prove the result by induction on d, that the only possibilities for the curve C are either to be rational of degree d or elliptic of degree $d+1$. Suppose the result true in $\mathbb{P}^{d-1}$, and let us consider a curve C in $\mathbb{P}^d$ of degree e and genus g; if it every d points on it span an hyperplane, then we can project away from a point p in C onto $\mathbb{P}^{d-1}$ and we get a curve C' of degree $e-1$, genus g, and again with the property that any $d-1$ points span an hyperplane (otherwise, together with p in $\mathbb{P}^d$ they would give d points on C not spanning an hyperplane in $\mathbb{P}^d$); hence, C' is either rational normal or elliptic normal, and so is C. To prove the other implication, supposing C contains d points not spanning an hyperplane, projecting away from one of these points gives a curve C' that cannot be rational or elliptic normal, hence neither can C. It only remains to prove the statement for in $\mathbb{P}^3$; we are going to use again the completeness of the adjoint system. Every d points spanning an hyperplane, in this case means that there are no trisecant lines; so, projecting C from a point p on the curve, we would get a curve C' in $\mathbb{P}^2$ of degree $e-1$ and genus g that has no double points, that means, that is smooth. In the same fashion as in the previous exercise, it is possible to prove that if we have a smooth plane curve C' of degree $e-1$ and genus g, p is any point of C', and $e-1 \geq 4$, then

$$h^0(\mathcal{O}_{C'}(1+p)) = 3$$

, that means that the curve does not come from a projection from a point on the curve in $\mathbb{P}^3$; hence, in our case we can conclude $e \leq 4$, so either the curve C in $\mathbb{P}^3$ is a twisted cubic, or an elliptic normal curve, or a rational quartic curve; but as we saw in Section **??**, the last one has indeed trisecant lines, hence it has to be excluded; this completes the proof. □

For the following three exercises, $C \subset \mathbb{P}^d$ will be an irreducible, nondegenerate curve and $2m-1 < d$. The exercises will prove the assertion made in the text that a general point on the m-secant variety $\mathrm{Sec}_m(C)$ lies on a *unique* m-secant $(m-1)$-plane to C.

Exercise 1.236. ?? Show, by a dimension count, that a general point of $Sec_m(C)$ lies on only *proper* secants; that is, $m-1$ planes spanned by m distinct points of C.

Solution to Exercise ??: Let us consider planes not coming from m distinct points; these lie in the image of the diagonal in $C^{(m)}$ (that is, an $m-1$-dimensional variety) in $\mathbb{G}(m-1,d)$; its universal family (the union of all such planes) cannot have dimension bigger than $2m-2$, so the general point of $\mathrm{Sec}_m(C)$ will not lie in these planes, because $\mathrm{Sec}_m(C)$ has dimension $2m-1$ (because curves are not defective). The only other possibility of points of $\mathrm{Sec}_m(C)$ that are not on proper secants, are points that lie on $m-1$-planes that contain m points of C that are not linearly independent; hence, that lie in the planes that we obtain when we take the closure of the image of the rational map $C^{(m)} \to \mathbb{G}(m-1,d)$; again, this is going to be a subscheme of $\mathbb{G}(m-1,d)$ of dimension at most $m-1$, so its total space inside $\mathrm{Sec}_m(C)$ will again be $(2m-2)$-dimensional, and it will not contain the general point of $\mathrm{Sec}_m(C)$. □

Exercise 1.237. ?? Using Lemma **??**, show that the variety of $2m$-secant $2m-2$-planes to C (equivalently, the locus $C_1^{(2m)}$ of divisors of degree $2m$ on C contained in a $(2m-2)$-plane) has dimension at most $2m-2$.

Solution to Exercise ??: □

Exercise 1.238. ?? Now suppose that a general point of $Sec_m(C)$ lay on 2 or more m-secant planes. Show that the dimension of the variety of $2m$-secant $(2m-2)$-planes to C would be at least $2m-1$.

Solution to Exercise ??: If a general point p lies in two m-secant $(m-1)$-planes, then considering the collection of $2m$ points, they span a $(2m-2)$-plane (because the two $m-1$-planes intersect), so the general point of $\mathrm{Sec}_m(C)$ lies on a $2m$-secant $(2m-2)$-plane, hence the locus of such planes is $(2m-1)$-dimensional; from the previous exercise, this is not possible. □

Exercise 1.239. ?? Show that if $C \subset \mathbb{P}^r$ is a general rational curve of degree d, and k is a number such that $d \geq r+k$ and $m-1 \geq k$, then the locus of m-secant $(m-k-1)$-planes has the expected dimension $m-k(r+1-m+k)$.

Solution to Exercise ??: A general rational curve in $\mathbb{P}^r$ of degree d is the same as the projection of a rational normal curve in $\mathbb{P}^d$ from a general $\mathbb{P}^{d-r-1}$; asking for the dimension of the locus of m-secant $(m-k-1)$-planes in $\mathbb{P}^r$ is the same as asking the dimension of the locus of m-secant $m-1$-planes in $\mathbb{P}^d$ that meet a general $\mathbb{P}^{d-r-1}$ in dimension at least $k-1$ (note that the condition $d \geq r+k$ is needed to have this situation actually happening); this is the same as intersecting in $\mathbb{G}(m-1,d)$ with a general Schubert cycle of type

$$\sigma_{\underbrace{(r+1-m+k,\ldots,r+1-m+k)}_{k}}$$

that has codimension in fact $k(r+1-m+k)$. □

Exercise 1.240. ?? By contrast with the preceding exercise, show that there exist components $\mathcal{H}$ of the Hilbert scheme of curves in $\mathbb{P}^3$ whose general point corresponds to a smooth, nondegenerate curve $C \subset \mathbb{P}^3$ with a positive-dimensional family of quadrisecant lines, or with a quintisecant line.

Solution to Exercise ??: Fixing 2 degrees d and e, complete intersections of surfaces of degrees d and e give general points of components of Hilbert schemes of curves in $\mathbb{P}^3$ (this is easy to prove, and it comes from the fact that all first order deformations of such a curve are still complete intersections). Then, if we consider $d=2$ and $e=4$, we get curves of type $(4,4)$ on quadric surfaces; the lines of the rulings of the quadric give 1-dimensional families of quadrisecant lines to these curves. Considering $d=3$ and $e=5$, we get curves on cubic surfaces; it is easy to prove that the 27 lines on the cubic surfaces are isolated quintisecant to these curves. Increasing e, we can get general points of Hilbert schemes of curves, with families of lines with arbitrarily high secancy order with given lines. □

Exercise 1.241. ?? Compute the number of quadrisecant lines to a general rational curve $C \subset \mathbb{P}^3$ of degree d. (Hint: in the notation of Section **??**, the answer is the degree of the class $\deg \tau^*(\sigma_{2,2}) \in A^4(\mathbb{P}^4)$. Express the class $\sigma_{2,2}$ in terms of the special Scubert classes σ_i and use (**??**) to evaluate it.)

Solution to Exercise ??: We can apply Exercise **??** in the case of $r = 3$, $m = 4$ and $k = 2$; we then need to find the intersection of the image of $C^{(4)} \cong \mathbb{P}^4$ in $\mathbb{G}(3, d)$ with a $\sigma_{2,2}$ cycle. From VERONESE SURFACE XXX STUFF, the class σ_1 pulls back to a WAIT FOR VERONESE. □

Exercise 1.242. ?? Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d and let g be the genus of a general hyperplane section of S; let e and f be the degrees of the classes $c_1(\mathcal{T}_S)^2$ and $c_2(\mathcal{T}_S) \in A^2(S)$. Find the class of the cycle $T_1(S) \subset \mathbb{G}(1, n)$ of lines tangent to S in terms of d, e, f and g. (Note: from Exercise **??**, we need only the intersection number $\deg([T_1(S)] \cdot \sigma_3)$; do this using Segre classes.)

Solution to Exercise ??: The number $\deg([T_1(S)] \cdot \sigma_3)$ we are looking for is the degree of the 4-dimensional variety swept out by all tangent lines to S; from Proposition **??**, this degree is the same as $\deg(s_3(\mathcal{E}))$; where $\mathcal{E}$ is the pullback of $\mathcal{S}$ from $T_1(S)$ to the source space, that is a $\mathbb{P}^1$ bundle over S; more in particular, this $\mathbb{P}^1$ bundle is $\mathbb{P}(\mathcal{T}S)$, because points of it are 1-dimensional subspaces of fibers T_pS. The Chow ring of this projective bundle is

$$A^*(\mathbb{P}(\mathcal{T}S)) = A^*(S)[\zeta]/(\zeta^2 + c_1\zeta + c_2).$$

The pullback $\mathcal{E}$ of $\mathcal{S}$ to $\mathbb{P}(\mathcal{T}S)$ containsa subbundle of the kind $\mathcal{O}_{\mathbb{P}^n}(-1)|_S$ (XXXXXbecause every tangent line contains one point of S) and the quotient will be

$$\mathcal{E}/\mathcal{O}_{\mathbb{P}^n}(-1)|_S \cong \mathcal{O}_{\mathbb{P}(\mathcal{T}S)}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)|_S$$

so for the Chern classes we have

$$c(\mathcal{E}) = (1 - H)(1 - H - \zeta)$$

so that passing to the Segre class we have

$$s_3(\mathcal{E}) = 4H^3 + 6H^2\zeta + 4H\zeta^2 + \zeta^3.$$

Note that H is the hyperplane class from the base S of the projective bundle, so that we have $H^3 = 0$, and $\deg(H^2\zeta) = d$. We then have

$$\deg(H\zeta^2) = \deg(-H \cdot c_1\zeta - H \cdot c_2) = -\deg(H \cdot c_1\zeta) = -\deg_S(H \cdot c_1).$$

We can then find this degree using adjunction on S and an hyperplane section, that says

$$\deg(H \cdot c_1) = 2g - 2 - d.$$

To find the degree of ζ^3, we can use twice the relation in the Chow ring, getting

$$\deg(\zeta^3) = \deg(-c_1\zeta^2 - c_2\zeta) = \deg(c_1^2\zeta + c_1c_2 - c_2\zeta) = \deg_S(c_1^2 - c_2) = e - f.$$

Summing everything up together, and using the result in Exercise **??** on $\deg([T_1(S)] \cdot \sigma_{2,1})$, we get

$$[T_1(S)] = (2d + e - f + 8g - 8)\sigma_{n-1,n-4} + (2d + 2g - 2)\sigma_{n-2,n-3},$$

that confirms what we found in Exercise **??**. □

Exercise 1.243. ?? Let $C \subset \mathbb{P}^3$ be a smooth curve of degree n and genus g, and S and $T \subset \mathbb{P}^3$ two smooth surfaces containing C, of degrees d and e. At how many points of C are S and T tangent?

Solution to Exercise ??: □

Exercise 1.244. ?? Show that the conclusion of Corollary **??** fails in characteristic $p > 0$:

(a) Let K be a field of characteristic 2, and consider the plane curve

$$C = V(X^2 - YZ) \subset \mathbb{P}^2.$$

Show that C is smooth, but that the dual curve $C^* \subset \mathbb{P}^{2*}$ is a line, so that $C^{**} \neq C$.

(b) Now let K be a field of characteristic p, set $q = p^e$ and consider the plane curve

$$C = V(YZ^q + Y^qZ - X^{q+1}) \subset \mathbb{P}^2.$$

Show that C is smooth, and that the dual curve $C^{**} = C$, but that $\mathcal{G}_C : C \to C^*$ is not birational!

Solution to Exercise ??: For a plane curve (or more in general for every hypersurface) the map into the dual is given by the partial derivatives, in particular in the first case the dual map is

$$[X, Y, Z] \to [0, -Z, -Y]$$

restricted to the curve C, that has the line $X' = 0$ (we will denote by X', Y', Z' coordinates of the dual space) as image: this is then the dual curve C^*; hence, C^{**} is just a point, definitely different from C. In the second example, the rational map is

$$[X, Y, Z] \to [-X^q, Z^q, Y^q].$$

We have

$$Y'(Z')^q + (Y')^qZ' - (X')^{q+1} = Z^qY^{q^2} + Z^{q^2}Y^q - X^{q(q+1)} = (Y^qZ + YZ^q - X^{q+1})^q = 0$$

so that the dual curve has equation $Y'(Z')^q + (Y')^q Z' - (X')^{q+1}$, that is the same as the one of C; repeating this process once again, we get $C^{**} = C$, but the duality map is the Frobenius morphism (composed with itself e times) so it is not birational, but it has nonreduced fibers of degree q. □

Exercise 1.245. ?? We saw in Section **??** that if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d > 1$ then the dual variety $X^* \subset \mathbb{P}^{n*}$ must again be a hypersurface. Show more generally that if $X \subset \mathbb{P}^n$ is any smooth complete intersection of hypersurfaces of degrees $d_i > 1$ then X^* will be a hypersurface.

Solution to Exercise ??: □

Exercise 1.246. ?? Let $X \subset \mathbb{P}^n$ be a k-dimensional *scroll*, that is, a variety given as the union

$$X = \bigcup \Lambda_b$$

of a one-parameter family of $(k-1)$-planes $\{\Lambda_b \cong \mathbb{P}^{k-1} \subset \mathbb{P}^n\}$; suppose that $k \geq 2$.

(a) Show that if $H \subset \mathbb{P}^n$ is a general hyperplane containing the tangent plane $\mathbb{T}_p X$ to X at a smooth point p then the hyperplane section $H \cap X$ is reducible; and
(b) Deduce that $\dim X^* \leq n - k + 2$ when $k \geq 3$.

Solution to Exercise ??: Consider the conormal variety $C(X)$ in $\mathbb{P}^n \times \mathbb{P}^{n*}$, that has (always) dimension $n-1$. Let us prove that fibers of the projection $C(X) \to X^*$ are positive dimensional, that means, hyperplanes that contain a tangent plane to X contain indeed a positive dimensional family of tangent planes (more precisely, we need to prove that fibers are at least $(k-3)$-dimensional). Remember that an hyperplane H contains the tangent plane $T_p X$ if and only if the hyperplane section $H \cap X$ is singular at p; now, if H is tangent to X at p, then it contains the entire $(k-1)$-plane Λ_p of the scroll where p lies in (because the tangent space contains all lines through p contained in X), and hence the hyperplane section is reducible (because X has degree $d > 1$, hence the hyperplane section too, so it has other components of total degree $d-1$). Now, inside the k-dimensional variety X, we have Λ_p and the residual section Γ; these are both codimension 1, so either they are disjoint or their intersection has codimension 2 or 1 in X; but they intersect at p (because their union is singular at p), so their union is indeed singular in a locus of dimension at least $k-2$; hence, the hyperplane H contains a $(k-2)$-dimensional variety of planes, and so the dual variety has dimension $(n-1)-(k-2) = n-k+1$. □

Exercise 1.247. ?? This is a sort of partial converse to Exercise **??** above. Let $X \subset \mathbb{P}^n$ be any variety. Use Theorem **??** to deduce that if the dual X^* is not a hypersurface, X must be swept out by positive-dimensional linear spaces.

Solution to Exercise ??: Suppose X^* is not an hypersurface, and hence that the general fiber of $C(X) \to X^*$ is positive dimensional (remember that all fibers of this map are linear spaces, because it is true for $C(X) \to X$, and by reflexivity also for X^*). But fibers of this map are subvarieties of X that have an hyperplane whose tangent spaces at all points are contained in a given hyperplane, and these fibers are linear spaces! SO X itself is covered by positive dimensional linear spaces. □

Exercise 1.248. **??** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that the dual variety X^* is necessarily singular.

Solution to Exercise ??: Suppose X is an hypersurface such that X^* is smooth, that means that the Gauss map $\mathcal{G} : X \to X^*$ is an isomorphism. Let us now consider a general net $\mathcal{B}$ of hyperplane sections of X, and its discriminant curve Δ; consider also the curve $\tilde{\Delta} \subset X \times \mathcal{B}$ of singular points of elements of the net. From considerations as in the end of Section **??** (see also Exercise **??**) we get that $\tilde{\Delta}$ is a curve with arithmetic genus

$$g(\tilde{\Delta}) = \frac{d(d-1)^2(dn - d - 2n + 1) + 2}{2}$$

and that the degree of Δ is $d(d-1)^{n-1}$ (that is just the degree of the dual hypersurface). Now, it is easy to see that if $d > 2$, the arithmetic genus of Δ is way bigger than the one of $\tilde{\Delta}$, so in the projection $\tilde{\Delta} \to \Delta$ there are some double points; but this means that some elements of the net are singular at more than point, and hence that we have hyperplanes that are tangent at more than one point, that means, that the Gauss map cannot be an isomorphism. If $d = 2$, on the other hand, hyperplane sections are either smooth quadrics in $\mathbb{P}^{n-1}$, or cones over a smooth quadric in $\mathbb{P}^{n-2}$, so that we do not have bitangent hyperplanes, and the Gauss map is indeed an isomorphism. □

Exercise 1.249. **??** Let $X = \mathbb{G}(1,4) \subset \mathbb{P}^9$ be the Grassmannian of lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by the Plücker embedding. Show that the dual of X is projectively equivalent to X itself!

Solution to Exercise ??: Remember that at a point $[L]$ of $\mathbb{G} := \mathbb{G}(1,4)$, we have

$$T_{[L]}\mathbb{G} \cap \mathbb{G} = \Sigma_2(L)$$

that means, all lines meeting L; if an hyperplane H cuts on $\mathbb{G}(1,4)$ hyperplane sections of type $\Sigma_1(P)$ where $P \supset L$ is a 2-plane, will contain it, so that H is tangent to $\mathbb{G}$ at all points $[L]$ where L is a line contained in P; this means that the projection $C(\mathbb{G}) \to \mathbb{G}^*$ has fibers of dimension at least 2, $\mathbb{G}^*$ has dimension at most 6 (and of course it is irreducible). Now, all hyperplanes cutting on $\mathbb{G}$ a section $\Sigma_1(P)$ are tangent to $\mathbb{G}$, and they constitute a 6 dimensional irreducible subvariety of X^*; hence, this is the entire X^*, that is hence isomorphic to $\mathbb{G}(2,4)$, that is isomorphic (and also linearly equivalent) to $\mathbb{G}(1,4)$. □

Exercise 1.250. **??** Let $X \subset \mathbb{P}^n$ be a smooth curve, and for any $k = 1, \ldots, n-1$ let

$$\nu_k : X \to \mathbb{G}(k, n)$$

be the map sending a point $p \in X$ to its osculating k-plane. Show that the tangent line to the curve $\nu_k(X) \subset \mathbb{G}(k, n)$ at $\nu_k(p)$ is the (tangent line to the) Schubert cycle of k-planes containing the osculating $(k-1)$-plane to X at p and contained in the osculating $(k+1)$-plane to X at p—in other words, to first order the osculating k-planes move by rotating around the osculating $(k-1)$-plane to X at p while staying in the osculating $(k+1)$-plane to X at p.

Solution to Exercise ??: □

Exercise 1.251. **??** If E is a smooth elliptic curve (over an algebraically closed field this means a curve of genus 1 with a chosen point), the addition law on E expresses the k^{th} symmetric power E_k as a $\mathbb{P}^{k-1}$ bundle over E. Verify this, and use it to give a description of $A(E_k)$.

Solution to Exercise ??: Let us pick $p_1, \ldots, p_{k-1}$ distinct points on E; for every degree k divisor $D \in E^{(k)}$, there exists an unique point $\pi(D) \in E$ such that

$$D = p_1 + \ldots + p_{k-1} + \pi(D)$$

or equivalently,

$$\pi(D) = D - (p_1 + \ldots + p_{k-1}).$$

This gives a morphism $\pi : E^{(k)} \to E$, whose fibers are all projective spaces $\mathbb{P}^{k-1}$, because they are the spaces

$$\mathbb{P}(H^0(\mathcal{O}_E(p_1 + \ldots + p_{k-1} + p))).$$

□

Exercise 1.252. **??** Using the preceding exercise, find the degrees of the secant varieties of an elliptic normal curve $E \subset \mathbb{P}^d$.

Solution to Exercise ??: □

1.11 Chapter 11